THE MATHEMATICAL GAZETTE

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THE NEED FOR REFORM IN THE TEACHING OF MATHEMATICS.

BY N. R. C. DOCKERAY.

EVEN more than the politician, the educationist must be prepared for the probable trend of events; for on him lies the responsibility of training the next generation to face those events. When civilisation is passing through a period of stability, this responsibility is not very onerous; no doubt he will conceive it to be his duty to be continually seeking for new methods of improving his teaching, but the questions, what shall he teach, and upon what principles shall his teaching be based, hardly arise; or, if they be mooted, are easily answered according to the accepted tenets of the day. In such times as these, philosophers are unlikely to propound theories which upset the whole basis upon which the current society rests. It may indeed happen that some genius, half a century before his time, attacks the foundations, metaphysical and ethical, which underlie the opinions of the day, but few are likely to heed him until changing circumstances compel the attention of the multitude, and the pedagogue, though perhaps influenced intellectually, will be unwilling to jeopardise his security by pointing out defects in a social adjustment which at the time appears to be stable, or by putting into practice theories which may well undermine that stability.

Thus education in England remained practically unchanged during the two centuries prior to 1800. The system in operation, though in many respects manifestly defective, worked well enough. Ruling power was for the most part in the hands of the landowners and in general only their sons were given the benefits of such education as the schools and universities were able or willing to provide. At the same time, outstanding ability in the son of a yeoman or artisan or even of a labourer was seldom stifled; Shakespeare, Newton, Metcalf, Brindley, Telford, Rennie, Watt, and a host of others are witnesses of this. The system was not ideal, far too much

stress was laid on Latin and Greek and far too little on mathematics as an element of culture, but, taken in conjunction with the current social circumstances, it gave at least as much opportunity for the development of individual tastes and abilities as our present system. I would indeed be inclined to say that it gave more.

The industrial revolution did more than increase the material comforts at our disposal, it completely altered our standard of values. It concentrated attention on the "use" to which any branch of learning could be put, and by "use" was meant commercial or industrial usability. Thus during the nineteenth century our outlook became increasingly material, so that to-day practically all spiritual conscience has vanished. So dominated are we by the economic machine that every question that arises, whether it be the loss of life on the roads, the desirability of building a State Opera House, the preservation of the amenities of the countryside, or the curriculum suitable for secondary school education, is decided on purely commercial lines, and so depraved have we become that we do not even regard such a state of affairs as being (as in fact it is) the negation of all ethical principles. The effect of this is quite noticeable in our schools. One of the commonest questions propounded to teachers is, "But sir, what *use* is this?" The idea which prompts this question is that if the facts cannot be put to some material use, it is not worth while learning them. Such a question would have been unthinkable 150 years ago, even if a boy had had the courage to ask it. No boy would have dreamt of questioning the value of the education provided for him; or, if he did, he would not have had material value in mind. I need hardly say that the result of questioning the utility of an unrelieved diet of classics has not been by any means wholly harmful, but the change in our scale of values has led to a false and very harmful emphasis being laid on material and commercial value in deciding educational syllabuses. Worse than this, our education has become subject to the law of supply and demand. We supply that sort of teaching for which there is a demand, instead of that which is most likely to develop the highest part of our pupils' minds. In other words, we no longer educate, we train for a career. Of course, we never have provided an education in the true sense of the term. But before the industrial revolution we taught what we then (probably quite wrongly) thought it was best for boys to learn, now we teach what is most likely to enable them to earn their living.

I suppose we mathematicians ought to be thankful that the demand for engineers, actuaries, statisticians, architects, surveyors, and business men has forced the schools to adopt mathematics as the major item in their curricula. Well, let us be duly grateful, but let not our gratitude cloud our judgment. For we know well that, as a result of this demand, 90 per cent. of the schoolmaster's time is spent, not in teaching mathematics, but in trying to instil the most elementary parts of commercial arithmetic into the heads of boys who hardly know their multiplication table. What is glorified by

the name of mathematics is now (with the exception of English) the only subject which is learned by practically every secondary schoolboy in the country. This is quite definitely all wrong, and we ought not to tolerate such a state of affairs. It is true that we are bound to carry on in view of the necessity of getting boys through the school certificate, which is demanded by practically all business men as a condition prior to granting admission to their firms, but this only illustrates my contention that education has been sacrificed on the altar of Supply and Demand to the devil which industrialism has let loose.

But it is not my purpose merely to criticise the education of to-day. I have in mind rather the question of how it will require to be modified in view of the rapidly changing circumstances in which we are likely soon to find ourselves placed. Let us make a brief review of these circumstances.

As long ago as before the war it was recognised that England's position as the pre-eminent industrial nation could not long be held. But it was impossible at that time to foresee how rapid the decline of her industries would be. The advent of nations which have discarded the Adam Smith system of liberal economics has completely altered the situation, and is likely to accelerate the decline of English industry. The fact is that we have almost completely lost our foreign markets. Germany, which before the war was our best customer both for raw materials and for manufactured goods, now buys little from us, our trade with other European countries and with Latin America has also declined, and although there has been an increase of business with South Africa, this has not been nearly enough to make up the balance. Recent events in China have gone far to destroy our trade with the Far East, and there is no prospect of our ever regaining it. Our industries are, in fact, now dependent on the home country, the dominions and the colonies for their markets. Apart from India, however, the whole population of the rest of the Empire is only about twice the population of Great Britain, while the demand for manufactured goods per head is certainly less than half. It is doubtful if exports to India are capable of great increase. Unless therefore a vigorous campaign advertising British goods in Canada, Australia and South Africa is initiated (and such a campaign would certainly be very expensive and almost certainly entirely unsuccessful), we may say that the chief markets for our industries must in future be found within the British Isles. Such a state of affairs alone would entail a considerable diminution of industrial activity, and under our system of free economics is likely to lead to a serious financial slump. But there are other factors affecting the situation. The decline in fertility which has now been in progress for nearly fifty years involves a serious alteration in the age constitution of our population, and (apart from the introduction of some unforeseen and improbable factor leading to a change in fertility) there will be in the near future, and continuing for some years, a disproportionate number of old people. Now old

people are poor consumers, and therefore this provides a further reason for expecting an industrial recession. Moreover, any attempt to exploit the home market involves paying high wages, and, in consequence, curtailing hours of work (since no man desires goods if he is given insufficient leisure in which to enjoy them), and this restricts production and therefore profits.

We have then every reason to expect a very rapid decline in industrial activity, starting in the near future. A succession of more or less serious slumps is likely to attend this decline, and in consequence a business career will prove continually less and less attractive to the boys who leave our schools.

And how is this likely to affect our educational arrangements, and, in particular, the teaching of mathematics? Seeing that mathematics was introduced into school teaching as a result of the demand of expanding industrialism and is largely maintained by the exigencies of our commercial civilisation, is there not some danger that the impending contraction of industrial activity may go some way towards ousting mathematics from our syllabuses? I myself think that the danger is a very grave one. We must not forget that we have allowed our teaching of mathematics to be so dominated by commercial interests that many of the finer minds are repelled from the subject. Many an idealistic boy, who shows promise of being a good mathematician, will reject mathematics in favour of classics after he has finished his school certificate, regarding the former as having no human or aesthetic interest, and, as his judgment is largely based on the type of question set in that examination, it is not an unreasonable one. Moreover, this early prejudice is liable to continue into later life, since the true nature of mathematics is never understood but by those who have made a study of it; such a man therefore will tend to influence his sons against the inclusion of mathematics as a cultural element in their education, and the prejudice will be perpetuated. Is it not a fact that the best boys in a school almost all choose the classical rather than the mathematical sixth, the latter being largely made up of budding engineers and actuaries?

A similar state of affairs will almost certainly arise in regard to the teaching of science. The average man to-day regards science, not as a discipline of knowledge based on the passionate love of truth, but as the mere handmaid of industrialism, as a tool with which to exploit the credulity of the masses for profit. The screeching advertisements in the press, with their constant repetition of the cry "Science tells us that . . ." or "Try this new scientific . . .", have done so much to discredit the word "science" that few people to-day realise that the search for truth, and not the exploitation of the misfortunes of others for gain, is the object of the true scientist. Science has in this respect suffered much more severely than mathematics; so severely indeed that I am inclined to think that, as the revolt against industrialism strengthens, science will tend to become of secondary importance in the school syllabus; and it will be difficult to stem this tide.

In regard to the revolt against mathematics, however, which is unlikely to be so serious, we have, to some extent, the remedy in our own hands. But we must put it into operation without delay. For there can be no doubt that the exponents of the classical tradition will not hesitate to grasp their opportunity. We shall be faced with a revival of all the old arguments in which the "humanities" are extolled as the only education whereby the mind can reach its highest development, while mathematics is dismissed as being of merely commercial interest. I have known men, otherwise highly educated, who were quite convinced that mathematics was entirely opposed to all cultural interests. They would admit that it has its uses—a knowledge of arithmetic, they might say, is of value in business, for book-keeping, etc., and no doubt higher mathematics has its applications also, in engineering, for example—but for the development of aesthetic appreciation and the cultivation of taste, whereby men may be trained to use their leisure to the best advantage, the only possible education is a literary and classical one. Such men, if it is pointed out to them that mathematics plays a decisive part in the formulation of any complete philosophy, are frankly incredulous. They would admit that, in the time of Pericles and the century following it, mathematics formed a valuable part of education, but since then it has degenerated, and now, although a certain minimum is necessary for the conduct of life, this should not be exceeded except by those who are compelled to do so by the exigencies of their careers.

This point of view is not uncommon, and may well become less so. It will, in fact, be very difficult to divorce mathematics from the materialistic outlook with which it is now so generally associated. But the effort must be made. To achieve our purpose we must continually point out the cultural value of mathematics. A distinction must be made, *before the school certificate stage*, between mathematics in the cultural sense, and "practical" mathematics, and this distinction should be stressed more and more as the school certificate stage is reached and passed. In teaching cultural mathematics, questions involving money or practical matters should be but lightly touched on; emphasis should be laid on the history of mathematical thought, on the part which mathematics has played in the greatest of all civilisations, on the influence which the development of mathematics has had on philosophy, and through philosophy on the whole basis of common thought.

These remarks apply chiefly to those who are prepared to proceed at least as far as the higher certificate, and the object I have in view is to persuade such boys that mathematics is the equal of classics or literature in the general development of culture. It would be well for boys of this type if they could escape the certificate entirely, but that, I am afraid, is impossible as yet, though no doubt it will come before long. At the same time, this reorientation of aim should not be confined to the brightest classes; the less intelligent might also benefit by it. At the present time we spent so much time in driving

home rules and methods, formulae and theorems, that we seldom pause to think whether we are doing our pupils any good. I am convinced that in a large number of cases we are doing positive harm, and that this tendency might be reversed if we were to make some effort to humanise the subject.

I do not wish to make any detailed suggestions in regard to syllabus. Any attempt to do so would inevitably bring about my ears a volume of criticism with which I should feel little inclination to deal, and which would probably be beside the point I wish to make. Some general remarks are all that I would venture at this stage. I think that monetary arithmetic might be almost entirely discarded. Such matters as simple and compound interest, stocks and shares, and all those tedious questions involving complicated percentages about buying and selling could be easily dropped. Quite apart from the fact that the method of dealing with percentages which is taught in schools is the one which is *not* used in the business world (where profits are expressed as a percentage of sale price), it is this type of arithmetical work which has done more than anything else to bring mathematics into disrepute. With these might also go all questions on races, baths, and the hands of clocks, likewise complicated algebraic fractions and heavy transformations of formulae. Boys who are good at mathematics find little difficulty in mastering theorems and usually enjoy riders, but there are many who gain nothing by being forced to learn theorems which they do not understand, either intellectually or emotionally, or to be logical when their whole nature rebels against the restrictions of logic. The time saved by discarding these topics could be more usefully spent in talks on anything in which mathematics has played a vital part; in talks on the development of the idea of number, on integers, fractions, negative numbers, and irrationals, on some of the curious properties of primes, on cubic equations and the quarrel between Tartaglia and Cardan, on the influence of Newton in implanting the idea of causality into the whole fabric of European thought for over two centuries, on the fourth dimension, on the possibility of using other scales of notation, on astronomy, on whether an infinite number of points can fill a finite space, on Achilles and the tortoise, and the flying arrow; in short, on any conceivable subject which the boys may care to raise and which can be used to show how mathematics has helped to extend the boundaries of knowledge. Above all, in talks on the history of mathematics, told in the form of stories about the great mathematicians. Euclid, Archimedes, Fermat, Newton, Euler, Gauss and Einstein have played as great a part in the progress of human thought and activity as Marlborough, Nelson or Wellington, and their names should be equally familiar. What should we think of a teacher of English literature who forced his pupils to learn Lycidas, but never mentioned the name of Milton? In discussing the lives of these men, stress should be laid on the fact that their object was to increase our knowledge of the Universe. The fact that this knowledge might be used to extend our control over Nature

for material ends should be dismissed as unimportant. In no circumstances should the boys be examined in the subjects of any of these talks.

It would clearly be impossible to carry these reforms into practice in the schools without some assistance from the universities. I suggest the establishment at Cambridge of a Tripos to be called Mathematical Greats, for which students would be required to spend one or two years after taking Part III of the Mathematical Tripos in studying the influence of mathematical thought on the development of philosophy. It will be readily admitted that no one can pretend to be a competent philosopher (unless he accepts complete Solipsism) without a knowledge of symbolic logic, of the work of Peano, Russell and Whitehead, of the nature of continuity and the various aspects of infinity, both " actual " and " becoming ", of the theory of relativity, of the influence of Newton, Planck and Heisenberg on the notion of causality, and of a host of other matters wherein mathematics has a direct bearing on philosophy. Even the clever undergraduate who has passed Part III with distinction has usually only the vaguest ideas on such matters as these, and might spend a useful year in attending lectures which would help him to co-ordinate his ideas and to assign to mathematics the place which it should rightly hold. I believe that the establishment of such a Tripos, analogous to Greats at Oxford, would do more than anything else to restore to mathematics the prestige which to-day it has unfortunately lost.

It would certainly have a vital effect on the teaching of sixth forms at schools. It is beyond my power to foresee what these effects might be, but I think it likely that the teaching of mechanics at least would be completely revolutionised. This would be no bad thing, for mechanics is at present the dullest subject in the scholarship syllabus, and the questions set in it are almost ludicrous. Other branches of sixth form work would no doubt also be benefited, and an important effect would be the endowment of mathematical study with a purpose which is not always apparent to the scholarship candidate to-day.

It should be the glory of mathematics that the Universe obeys its laws. For it is at least partial evidence that mathematical thought is intrinsic in the mind of God. Even if we contend, with Eddington, that the concepts we have formed of Nature and her laws are mere reflections of the constitution of our own minds, the argument is not invalidated, so long as we admit that the human mind has a Divine origin or has at least some part in Divinity. But instead of being the glory of mathematics, this fact may prove to be its greatest misfortune. For we have exploited it to pander to the baser elements of human nature, the love of money and the desire for material comforts. The time has come to rescue mathematics from the depredations of its materialistic parasite, lest it be destroyed entirely.

N. R. C. D.

A PROBLEM IN CARTOPHILY

By F. G. Maunsell.

I SHOULD like to say straightaway that collecting cigarette-cards is *not* my hobby. But recently the manufacturers of the brand of cigarette I smoke started to issue an attractive series of cards and I said to a friend, "I think I shall save these cards until I obtain the complete set of fifty." He replied, "About how many packets do you think you will have to buy before you get the set?" And this raises an interesting problem in probability which I do not recollect having seen before.

Suppose the set is one of r different cards. We assume, of course, that the manufacturers put the cards into the packets at random. The problem is then equivalent to the following: "if r different balls are placed in an urn and one ball is drawn at a time and replaced after each drawing, what is the average number of drawings required for each ball to be drawn at least once?" Or again: "if a man takes p drawings as above to obtain the complete set of r balls and is then to receive p shillings, what is his expectation?"

We will first find the chance of obtaining the complete set of r objects in n draws *or fewer*, where n is greater than or equal to r. Call this chance $C(n, r)$.

Then

$$C(n, r) = P(n, r)/r^n,$$

the denominator being the number of permutations of r different objects taken n at a time, repetitions being allowed, and $P(n, r)$ the number of such permutations in which each of the objects occurs at least once. To find $P(n, r)$ we notice that the problem is equivalent to Example 6, p. 458 of Durell and Robson's *Advanced Algebra*,* the result being

$$r^n - \binom{r}{1}(r-1)^n + \binom{r}{2}(r-2)^n - \dots .$$

Thus

$$C(n, r) = r^{-n}\left\{r^n - \binom{r}{1}(r-1)^n + \binom{r}{2}(r-2)^n - \dots\right\}.$$

If we denote the chance of obtaining the complete set in *exactly* n draws by $D(n, r)$, then

$$\begin{aligned} D(n, r) &= C(n, r) - C(n-1, r) \\ &= r^{-n}\left\{r^n - \binom{r}{1}(r-1)^n + \binom{r}{2}(r-2)^n - \dots \right. \\ &\qquad \left. - \left(r^n - r\binom{r}{1}(r-1)^{n-1} + r\binom{r}{2}(r-2)^{n-1} - \dots\right)\right\} \end{aligned}$$

* Durell and Robson, *Advanced Algebra* (Bell & Sons, London, 1937), vol. iii, p. 458. The result is also given as an example in Chrystal's *Algebra* and is there attributed to de Moivre.

$$= r^{-n}\left\{r(r-1)^{n-1} - 2\binom{r}{2}(r-2)^{n-1} + 3\binom{r}{3}(r-3)^{n-1} + \ldots\right\}$$

$$= r^{-n+1}\left\{(r-1)^{n-1} - (r-1)(r-2)^{n-1} + \binom{r-1}{2}(r-3)^{n-1} - \ldots\right\}.$$

The required expectation, $f(r)$ (say), is given by

$$f(r) = \sum_{n=r}^{\infty} nD(n, r)$$

$$= \sum_{s=0}^{r-2}\left\{(-)^s\binom{r-1}{s}\sum_{n=r}^{\infty} n\left(\frac{r-s-1}{r}\right)^{n-1}\right\}.$$

Now $$\sum_{n=r}^{\infty} n\left(\frac{r-s-1}{r}\right)^{n-1}$$

$$= r\sum_{n=r}^{\infty}\left(\frac{r-s-1}{r}\right)^{n-1} + \sum_{n=r}^{\infty}(n-r)\left(\frac{r-s-1}{r}\right)^{n-1}$$

$$= r\left(\frac{r-s-1}{r}\right)^{r-1}\sum_{p=0}^{\infty}\left(\frac{r-s-1}{r}\right)^{p} + \left(\frac{r-s-1}{r}\right)^{r}\sum_{q=1}^{\infty} q\left(\frac{r-s-1}{r}\right)^{q-1}$$

$$= r\left(\frac{r-s-1}{r}\right)^{r-1}\left(1 - \frac{r-s-1}{r}\right)^{-1} + \left(\frac{r-s-1}{r}\right)^{r}\left(1 - \frac{r-s-1}{r}\right)^{-2}$$

$$= r\left(\frac{r-s-1}{r}\right)^{r-1}\left(\frac{s+1}{r}\right)^{-1} + \left(\frac{r-s-1}{r}\right)^{r}\left(\frac{s+1}{r}\right)^{-2}$$

$$= \frac{(r-s-1)^{r-1}}{(s+1)^2 r^{r-2}}\{(s+2)r - (s+1)\}.$$

Hence $$f(r) = r^{-r+2}\left\{(r-1)^{r-1}(2r-1) - (r-1)\frac{(r-2)^{r-1}}{2^2}(3r-2) + \binom{r-1}{2}\frac{(r-3)^{r-1}}{3^2}(4r-3) - \ldots\right\}, \quad \ldots(1)$$

the series having $r-1$ terms.

At first I thought it impossible to express $f(r)$ in a simpler form, and this series was used to compute $f(r)$ in a few simple cases. Later, however, I discovered a slightly simpler series for $f(r)$, obtained in the following manner.

$$f(r) = \sum_{n=r}^{\infty} nD(n, r)$$

$$= \sum_{n=r}^{\infty} n\{C(n, r) - C(n-1, r)\}$$

$$= \lim_{k\to\infty}\{rC(r, r) + (r+1)[C(r+1, r) - C(r, r)] + \ldots + (r+k)[C(r+k, r) - C(r+k-1, r)]\}$$

$$=\lim_{k\to\infty}\{-C(r,r)-C(r+1,r)$$
$$-\ldots-C(r+k-1,r)+(r+k)\,C(r+k,r)\}$$
$$=\lim_{k\to\infty} rC(r+k,r)-\lim_{k\to\infty} k\{1-C(r+k,r)\}$$
$$+\lim_{k\to\infty}\sum_{j=0}^{k-1}\{1-C(r+j,r)\},$$

provided these three limits exist.

Now $\lim_{k\to\infty} C(r+k,r)=1$, since this is the chance of obtaining the complete set in an infinite number of draws or less.

Hence $$\lim_{k\to\infty} rC(r+k,r)=r.$$

Also $$\lim_{k\to\infty} k\{1-C(r+k,r)\}$$
$$=\lim_{k\to\infty}\left\{k\binom{r}{1}\left(\frac{r-1}{r}\right)^{r+k}-k\binom{r}{2}\left(\frac{r-2}{r}\right)^{r+k}+\ldots\right\}=0,$$

since each term is of the form $Ck\theta^k$ where C is a constant and θ is less than unity, and there are a finite number of terms.

Finally, $$\lim_{k\to\infty}\sum_{j=0}^{k-1}\{1-C(r+j,r)\}$$
$$=\left[\binom{r}{1}\left(\frac{r-1}{r}\right)^{r}-\binom{r}{2}\left(\frac{r-2}{r}\right)^{r}+\ldots\right]$$
$$+\left[\binom{r}{1}\left(\frac{r-1}{r}\right)^{r+1}-\binom{r}{2}\left(\frac{r-2}{r}\right)^{r+1}+\ldots\right]+\ldots.$$
$$=\binom{r}{1}\left(\frac{r-1}{r}\right)^{r}\frac{1}{1-\frac{r-1}{r}}-\binom{r}{2}\left(\frac{r-2}{r}\right)^{r}\frac{1}{1-\frac{r-2}{r}}+\ldots$$
$$=\binom{r}{1}\left(\frac{r-1}{r}\right)^{r}r-\binom{r}{2}\left(\frac{r-2}{r}\right)^{r}\frac{r}{2}+\ldots.$$

Thus

$$f(r)=r\left\{1+\binom{r}{1}\left(\frac{r-1}{r}\right)^{r}-\binom{r}{2}\left(\frac{r-2}{r}\right)^{r}\frac{1}{2}+\binom{r}{3}\left(\frac{r-3}{r}\right)^{r}\frac{1}{3}-\ldots\right\}\ldots.(2)$$

It has seemed worth while to give both formulae for $f(r)$ as the equality of (1) and (2) makes an interesting identity. The second expression would have been slightly easier for computation, although neither is very good.

The interesting question remains as to what is the order of magnitude of $f(r)$ for large r. It seems very probable that

$$f(r)\sim r\log r.$$

The further problem arises as to what is the *most probable number* of cards a collector will collect in obtaining his set. This number, which I shall call $\phi(r)$, is not the same as $f(r)$, the occasional occurrence of very high numbers increasing $f(r)$ but leaving $\phi(r)$ unaffected. In statistical language $f(r)$ is the "mean" and $\phi(r)$ the "median".

In order to find the value of $\phi(r)$ we have to find a number such that the chance of obtaining the set in fewer cards is equal to the chance of obtaining the set in more cards. That is, we must find n so that $C(n, r)=\frac{1}{2}$. In general, the value of n so obtained will not be an integer, and I have confined myself to finding the value of $[\phi(r)]$, the greatest integer not greater than $\phi(r)$. This is comparatively simple as, when r is fairly large, only a few terms have to be taken in order to find whether $C(n, r)$ is less or greater than $\frac{1}{2}$.

It is possible to prove that $\phi(r)\sim r \log r$, and I hope to publish this proof in a subsequent paper. It is this result that makes it plausible that $f(r)\sim r \log r$ also, since commonsense would seem to indicate that the ratio of $f(r)$ to $\phi(r)$ will tend to unity as r tends to infinity.

The following table gives the values of $f(r)$, $[\phi(r)]$, and $r \log r$ in a few simple cases (correct to one decimal place in the cases of $f(r)$ and $r \log r$).

r	$f(r)$	$[\phi(r)]$	$r \log r$
2	3	2	1·4
3	$5\frac{1}{2}$	4	3·3
4	$8\frac{1}{3}$	6	5·5
5	11·4	9	8·0
6	14·7	12	10·8
10	29·3	26	23·0
25	95·3	89	80·4
50	225·0	213	195·6

A few rough checks have been made. For instance, 100 trials with r equal to 10 gave $f(r)$ as 29·65, which is fairly near the theoretical value. In this check the last digits in Chambers' seven-figure tables were used as the "cards". F. G. MAUNSELL.

GLEANINGS FAR AND NEAR.

1196. How does the idea of God fare with these modernist thinkers? If He exists at all, He is organic with the world, which is as necessary to Him as He is to it. He appears under strange names—Space—Time, . . . , Duration, . . . A mathematician, it has been suggested, might pray to x^n.—W. R. Inge, *A Rustic Moralist*, p. 6. [Per Mr. A. F. Mackenzie.]

"A WEIGHTY MATTER."

By G. H. Grattan-Guinness.

Through no fault of the administrators of the Association my October number of the *Gazette* has come to hand five months late, and with it the letters of Mr. Tuckey and Mr. Snell in criticism of my article, "A Weighty Matter" (*Gazette*, May 1937). On account of pressure of other affairs, the absence of the missing number was overlooked; I trust that the gentlemen concerned have not assumed that I have succumbed to the adverse portions of their comments "hands down", and hasten to assure them to the contrary as well as to apologise to them for apparently ignoring their replies. I am grateful for the opportunity afforded me by the publication of their open letters to reconsider not so much the *substance* as the *wording* of certain statements in my article, and at the same time to bring forward other suggestions and comments which I have received from other correspondents.

Firstly, I beg leave to ignore any arguments "following Einstein", for, as I said at the beginning of my article, I am concerned with a procedure suitable for youngsters of eleven years of age, beginning their secondary school careers. For the same reason, I reject the fancy of a balance whose arms are so long that the value of g at one end is distinguishable from that at the other; it would be quite unintelligible to such pupils, and, in any case, I doubt if it be anything more than a figment of Mr. Tuckey's fertile imagination. To mention but three objections: if such an instrument were to be constructed, (*a*), what about the mere sag in arms of such length; (*b*), would the arms have to follow the curvature of the earth; and (*c*), assuming (*b*) granted, would the same curvature be suitable for both equatorial and meridianal use, and, along the same meridian, at different latitudes?

Secondly, while I appreciate Mr. Snell's contention that the definition of mass should be left till late on in the course, and in my article agreed to a late revision of the matter from a dynamatical standpoint, I advocate a need for an early conception. To my mind, it is absurd to attempt to deal, for example, with momentum ($\equiv$ product of units of mass and units of velocity), as Mr. Snell seems to suggest, before the mass on which it depends. Besides, even if we as teachers of Mathematics do not do so, the Science teachers will quite early be after density in reference to various things, and, unless the mass question has been treated beforehand, we are going to be up against its quite erroneous definition of "weight per unit volume", the sort of thing which it is my aim to avoid at all costs. Let a child use a misconception for three or four years, and then try to unteach him. . . .

While all my correspondents appear to agree with the main development of my theme, their disagreements seem to be, as in Mr. Snell's case, with my definition ("wrong and misleading") of

mass-measurement by the balance. Mr. Tuckey suggests that, according to what I wrote in § 2.2, one might as well measure smells by that means. I think the trouble has arisen over a faulty wording : in trying to abbreviate, I confused my original meaning with another.

If a balance does not measure mass, what else does it measure? The general trend of opinion seems to be that it compares weights. Now, I do not disagree with this (of course), but I would emphasize the following points :

(1) Suppose a given body to be placed in a scale pan of an equal-armed accurate balance, and to be balanced by a certain number of standard unit " pounds " (exact copies, shall we imagine, of the British Imperial Pound itself?) in the other pan. Suppose, further, that the whole affair be removed to another region, on the earth, on the moon, or anywhere in the universal gravitational field. Result : still equilibrium. Yet the *weights*, W_1 and W_2 " somethings ", on both sides of the balance have changed to new values, W_1' and W_2', and all that has been obtained is the fact that, if $W_1 = W_2$ at first, $W_1' = W_2'$ now, where $W_1' : W_1 = W_2' : W_2 =$ the ratio of the two values of g in the two regions. One has *compared* the weights, certainly ; as regards direct *measurement*, one has merely measured a variable weight with a variable standard. To give an analogy in terms of heat : it should, I imagine, be an easy matter to construct a vessel of suitable shape so that, when filled with a certain liquid at a certain temperature equal to that of the vessel itself, the expansion of the latter, degree by degree, would, to a very close approximation, just counteract the corresponding expansion of the liquid. Thus, it would appear that the original pint (say) of liquid remained a pint throughout the given range of temperature. All that has been done, of course, is to establish that, if $V_1 = V_2$ at first, then $V_1' = V_2'$ afterwards, where $V_1' : V_1 = V_2' : V_2 = (1 + \alpha t') : (1 + \alpha t)$. A variable volume " measured " with a variable unit ; *compared* certainly—but *measured*?

(2) If this is admitted, must a balance be condemned to give no direct measurement at all? Why should not the mass of a body (yes, Mr. Tuckey, a *property* of the body, I agree ; I used the phrase " equal masses " purely as an abbreviation of " bodies with equal masses "), as regards its measurement, be *defined* as that which is measured on a balance? To revert to my offending paragraph with Mr. Tuckey's " amendment " :

> " The two objects have been made identical : they must therefore have equal masses—volumes—colours—shapes—even smells (whatever we mean by any of these), for being identically equal they are equal in every respect. Thus equal *somethings* will balance. . . ."

Now we already know how to measure volumes—in my sequence of units, length, area, volume and capacity precede mass and weight : they are much easier to comprehend. We have, in addition, optical instruments, including our eyes, to measure shape and colour ; for

all I know, there may be smell-measuring instruments (there are certainly noses which at least act as " smelloscopes " even if they be limited in their action as smellometers!). And we know that our ways of measuring these properties have nothing to do with balancing. So with all the other fundamental physical properties of the bodies—except their masses and their weights. Why should we not say?—" There is obviously something about these two objects that the balance is registering. Admitting that it is not their weights (save by comparison), let us call it their masses." This is, be it emphasized, frankly a definition ; what we measure with a foot-rule we call the length of a body, and, *in extenso*, its surface area, its volume ; what we measure with a pint pot we call its volume in the case of a liquid ; what we measure by this, that, and the other method we call its this, that, and the other property ; what we measure on the balance we call its mass : what we measure on the dynamometer we call its weight. Why not?

To put it another way :

(*a*) Is it granted that, in the gravitational field, which is all that I am concerned with, a body's mass remains unaltered from place to place?

(*b*) Is it granted that its weight varies?

(*c*) Is it granted that a balance gives a constant reading for a given body?

(*d*) Is it granted that a dynamometer gives a variable reading and is directly subject to the weight (*i.e.* earth-pull, moon-pull, etc., according to the locality) of the body?

Then it must be admitted that a balance cannot *measure* weight or a dynamometer mass ; on the other hand, why should not the former measure mass and the latter weight?

(3) The answer as given by Mr. Tuckey is that " the only definition or explanation which we can give of ' mass ' " is " suggested by the phrase ' inverse ratio of accelerations'." The *only* one? I cannot see why the dynamical school should stand up and declare its views to be, as it were, inspired by heaven. There is, in my estimation, a statical view as well, on the lines I have indicated, and, if its source be " the other place ", I must needs be content to accept the fact! To argue that the one is right while the other is wrong is, to my mind, as sensible as to assert that, of the two conceptions of integration (inverse of differentiation and area-summation) one is erroneous. These two points of view are but different aspects of the same thing, and can be shown to be quite compatible, one with the other. So, surely, is this question of mass. If a pound of, say, sausage is measured, by a sort of Fletcher's trolley method with two identical vehicles, one of which contains a standard pound measure and the other so much sausage that the two suffer equal accelerations under the action of equal external forces—this, I take it, is the sort of thing Mr. Tuckey has in mind when he says that the word " mass " must be replaceable by the word " inertia "—then surely these two

substances, standard and sausage, must have equal weights, for they have equal gravity accelerations in the same region. If their weights are equal, then, by moments, they will balance on an ordinary pair of scales, and will thus register equal masses according to the system I am advocating. In other words, whether we regard mass statically or dynamically, whether we measure it on a balance or on a Fletcher's trolley, we come back to the same quantity of matter in each case ; it is of no consequence whether we start with one or the other system.

Mr. Snell rightly regards the (dynamical) concept of mass as being hard, to be introduced gradually and late. But, to sum up, my contention is that :

(1) there is another, statical, concept, via the balance, which is quite compatible with the other, at least in any ordinary gravitational field and therefore in all elementary mechanics;

(2) this statical concept is easy enough for a young pupil ;

(3) it can be used as a basis of a sequence which, while necessarily requiring big assumptions, such as Newton's Law of Gravitation, certainly does not involve any definite " volte face " at a later stage ; all that happens is something rather closely akin to the modern methods of tackling Geometry, an introductory stage with several gaps in the logical sequence before a much later revisionary course as complete as possible—and this revisionary (dynamical) course, having been shown to be consistent with the preliminary (statical) so far as our present gravitation is concerned, may be extended, if desired, to include cases of variant masses (" following Einstein "), other types of (even complete screening from) gravitation—Mr. J. T. Combridge raised objections on these grounds to the statical concept—and anything else that the mind of man can devise.

Whatever opinions one may hold of the rightness or otherwise of my proposals, and whatever official decision (if any) be finally reached concerning the matter, there seems to me, from correspondence I have received, no doubt that the *confusion* which arises between mass and weight is not merely due to a philosophical difficulty but is liable to persist, even when that has been overcome, on account of the similarity in nomenclature of units, " pound " and " pound-weight ", " gram " and " gram-weight ", etc. I have strongly advocated a rigorous discrimination from the very beginning between these two, and have deplored the tendency of certain people to use the one term " pound " or " gram " for the measurement of both properties. I think I am safe in feeling that the general consensus of opinion will be with me here ; I have actually received indication that some people would even go further—Mr. Fairthorne for example, reminds us (*Gazette*, Feb. 1938, p. 77) of the aeronautical use of the admirable term " slug " for measurement of mass, though, from what he says, I gather that a slug is equivalent to g pounds mass, and therefore is not an *absolute* unit of measurement at all,

being dependent on a variable g (or, if you like, on a variable unit of force, " pound-weight "). The objection is, of course, that mass, being an invariant property of a body, should surely be measured by an invariant unit ; in the case of weight, however, the same thing does not necessarily apply.

Mr. J. K. D. Musgrave, writing to me some time ago, said : " To me it seems the only thing is to remove that ambiguous word ' weight ' from all text-books in reference to the word ' pound '. Why not g poundals = 1 lib?—a ' lib ' is definitely a force. I have tried it with some youngsters and they made a great game of it. Why should we be handicapped because the tradesmen use a word in its unscientific sense? " This appeals to me as being a most valuable suggestion, though as " lb. " is, of course, an abbreviation for " libra ", I think a different spelling would be better ; would a final consonant " g ", indicative of the gravity aspect of the affair, meet the case? A term such as " lig " for unit force, corresponding to " lb. " for unit mass, would avoid all the clumsiness of adding the word " weight " to our units, and would be suitable for all " practical " problems involving smallish quantities. This would overcome Mr. Snell's objection that the definition of mass given by me " would lead the boy to talk of the force due to the mass of the load " (*i.e.* on a lever) " and would give him an idea of mass which has later to be undone ". If the said boy is properly taught, he will, of course, refer to nothing of the sort ; he will talk of the force due to the earth's pull on the load ; he will hang a load whose mass is, say, 10 lb. on the end of his lever, but, in drawing his arrow, he will attach a label of " 10 lig ", not " 10 lb.", even if Mr. Snell cannot persuade him to add the word " wt." under the present system!

The poundal can still remain, if desired ; stones, quarters, hundredweights, and any corresponding force units can be omitted, the whole lot being reduced to " lb." or " lig ", while for large units, in engineering problems on trains, etc., the " ton " will measure mass, as usual, and for force, why not the " tog "? Metrically we have the " gram ", and so why not the " grag ", with the usual multiples and submultiples? Alternatively, it seems rather a pity to throw away the chance of a metric unit so explicit as the " grav ".

Finally, regarding the tradesman problem, where what the scientist calls " mass " he names " weight ", could not this be overcome by giving to " mass " and " weight " identical meanings, merging them into the unity of what we now call " mass ", and inventing a new term for the present scientific " weight "? The very permanence of the name " hundredweight "—to say nothing of " pennyweight "—as a measure of mass is a strong argument in favour of the merger. And, for the new term for scientific weight or earth-pull, what about " tug "? Is there anything too awful to be considered in the suggestion that we might say that the pound is the measurement of the mass or weight (according to our political or religious convictions) of a certain standard lump of material and that the lig is the measurement of its tug? " A litre of water has a

mass or weight of one kilogram and a tug of one kilograv." No worse, surely, than the bole, the sproud, and other things which it was attempted to introduce as certain mechanical units? Such a procedure might satisfy all demands and all circumstances: the confusion between mass and weight would dissolve in an identity, from which, phoenix-like, would rise a new creation for gravitational force; we could then permit ourselves to take weights out of our weight boxes with impunity, and Mr. Tuckey could wind up the dittoes of his clock with a clear conscience!

G. H. GRATTAN-GUINNESS.

1197. DIALECTICS IN MATHEMATICS.

"It is the same in mathematics. Let us take any algebraical magnitude whatever: for example, a. If this is negated, we get $-a$ [minus a]. If we negate that negation by multiplying $-a$ by $-a$, we get a^2, *i.e.* the original positive magnitude, but at a higher degree, raised to its second power. In this case also it makes no difference that we can reach the same a^2 by multiplying the positive a by itself, thus also getting a^2. For the negated negation is so securely entrenched in a^2 that the latter always has two square roots, namely, a and $-a$. And the fact that it is impossible to get rid of the negated negation, the negative root of the square, acquires very obvious significance as soon as we get as far as quadratic equations. The negation of the negation is even more strikingly obvious in the higher analyses, in those "summations of indefinitely small magnitudes" which Herr Dühring himself declares are the highest operations of mathematics, and in ordinary language are known as the differential and integral calculus. How are these forms of the calculus used? In a given problem, for example, I have two variable magnitudes x and y, neither of which can vary without the other also varying in a relation determined by the conditions of the case. I differentiate x and y, *i.e.* I take x and y so infinitely small that in comparison with any real magnitude, however small, they disappear, so that nothing is left of x and y but their reciprocal relation without any, so to speak, material basis, a quantitative relation in which there is no quantity. Therefore, dy/dx, the relation between the differentials of x and y, is equal to $0/0$, but $0/0$ as the expression of y/x. I only mention in passing that this relation between two magnitudes which have disappeared, caught at the moment of their disappearance, is a contradiction; it cannot disturb us any more than it has disturbed the whole of mathematics for almost two hundred years. And yet what have I done but negate x and y, though not in a way that I need not bother about them any more, not in the way that metaphysics negates, but in the way that corresponds with the facts of the case? In place of x and y, therefore, I have their negation, dx and dy, in the formulae or equations before me. I continue then to operate with these formulae, treating dy and dx as magnitudes which are real, though subject to certain exceptional laws, and at a *certain point I negate the negation*, *i.e.* I integrate the differential formula, and in place of dx and dy again get the real magnitude x and y, and am not then where I was at the beginning, but by using this method I have solved the problem on which ordinary geometry and algebra might perhaps have broken their teeth in vain."—From Friedrich Engels, *Herr Eugen Dühring's Revolution in Science* (1877), as quoted in Emile Burns [editor], *A Handbook of Marxism*, pp. 263-5. Eugen Dühring's own views on this are not available here, but as he is known to have had an unpleasant controversy with Helmholtz, and to have attracted large student audiences, those views might be equally worth gleaning. [Per Mr. D. D. Kosambi.]

A CLASS OF DIOPHANTINE EQUATIONS.

By R. L. Goodstein.

The general solution in positive integers of the equation

$$2a(x^2-y^2)+1=z^2 \tag{1}$$

where a is a prime number (not 2).

Writing the equation in the form

$$z^2=\{a(x^2-y^2)+1\}^2-\{a(x^2-y^2)\}^2,$$

we have

$$a(x^2-y^2)+1=\alpha^2+\beta^2,$$

$$a(x^2-y^2)=2\alpha\beta.$$

Therefore $\alpha=\beta+1$ and $z=2\beta+1$.

Hence $a(x-y)(x+y)=2\alpha\beta=2\beta(\beta+1)$;

put $x-y=\eta$, then

$$a\eta(\eta+2y)=2\beta(\beta+1).$$

Hence $\eta(\eta+2y)$ is a multiple of 2 (for a is prime to 2).

Thus η is even. Put $\eta=2\xi$, and we have

$$2a\xi(\xi+y)=\beta(\beta+1)=2\kappa(2\kappa\pm1),$$

i.e.

$$a\xi(\xi+y)=\kappa(2\kappa\pm1).$$

Hence $\kappa(2\kappa\pm1)$ is divisible by a, so that either κ or $2\kappa\pm1$ is divisible by a (since a is prime).

Thus either $\kappa=\lambda a$ or $\kappa=\dfrac{(2\mu+1)a\pm1}{2}$

(a is odd, so that κ is an integer).

Therefore either

$$\xi(\xi+y)=\lambda(2a\lambda\pm1), \tag{1.01}$$

or

$$\xi(\xi+y)=(2\mu+1)[(2\mu+1)a\pm1]/2. \tag{1.02}$$

Consider (1.01). Let $\lambda=jk$, and let l, m be an ordered factor pair of the integer $(2ajk\pm1)$ $\{i.e.\ lm=2ajk\pm1\}$,

then

$$\xi=jl,$$

$$\xi+y=km;$$

whence $y=km-jl$ and $x=y+\eta=y+2jl=km+jl$

and $z=2\beta+1=4\kappa\pm1=4ajk\pm1$.

Thus one set of solutions is

$$\begin{cases} x=km+jl, \\ y=km-jl, \\ z=4ajk\pm1. \end{cases} \tag{1.03}$$

Consider (1.02). Let $2\mu+1=(2j+1)(2k+1)$ and let p, q be an ordered factor set of the integer $\{(2j+1)(2k+1)a\pm1\}/2$.

Then the second set of solutions is

(1.04)
$$\begin{cases} x = p(2j+1) + q(2k+1), \\ y = p(2j+1) - q(2k+1), \\ z = 2a(2j+1)(2k+1) \mp 1. \end{cases}$$

Notice that in (1.03) the value of z is $4ajk+1$ if l, m is a factor pair of $2ajk+1$, and is $4ajk-1$ if l, m is a factor pair of $2ajk-1$; whereas in (1.04) the ambiguous sign in the value of z is *opposite* to that in the integer $\{(2j+1)(2k+1)a \pm 1\}/2$.

When $a=1$, the two sets of solutions (1.03) and (1.04) coincide, so that the general solution of

(1.1) $$2(x^2-y^2)+1=z^2$$

(1.11) is
$$\begin{aligned} x &= km+jl, \\ y &= km-jl, \\ z &= 4jk \pm 1, \end{aligned}$$

where j and k are any integers and l, m is a factor pair of the integer $2jk \pm 1$.

The case when $a=2$ is included in the following equation.

The general solution of the equation

(1.2) $$4a(x^2-y^2)+1=z^2,$$

where a is prime (not 2), may be obtained by the same method.

The general solution is given by the three sets :

(1.21)
$$\begin{aligned} x &= km+jl, \\ y &= km-jl, \\ z &= 8ajk \pm 1, \end{aligned}$$

where j and k are any integers and l, m is a factor pair of the integer $4ajk \pm 1$;

(1.22) and
$$\begin{aligned} x &= p(4j \pm 1) + q(4k \pm 1), \\ y &= p(4j \pm 1) - q(4k \pm 1), \\ z &= 8r + (-1)^{n+1}, \end{aligned}$$

where $r = \{(4j \pm 1)(4k \pm 1)a + (-1)^n\}/4$, the ambiguous signs being either both positive or both negative and n being odd or even according as a is of the form $4t+1$ or $4t-1$; p, q is a factor pair of the integer r ;

(1.23) and
$$\begin{aligned} x &= p'(4j \pm 1) + q'(4j \mp 1), \\ y &= p'(4j \pm 1) - q'(4j \mp 1), \\ z &= 8s + (-1)^{m+1}, \end{aligned}$$

where $s = \{(4j \pm 1)(4k \mp 1)a + (-1)^m\}/4$, the ambiguous signs being opposite, and m odd or even according as a is of the form $4t-1$ or $4t+1$; p', q' is a factor pair of the integer s.

(1.24) When $a=1$, the three sets of solutions coincide.

The general solution of the equation

(1.3) $$2a(x^2-y^2)+1=c^2z^2,$$

where a is prime and c is any integer, may be deduced from the solution of (1).

Let $x=g_t,\ y=h_t$ be the general solution of the linear equation $cx-4ay=1$ and $x=g_t',\ y=h_t'$ that of $cx-4ay=-1$.

And similarly let $x=\gamma_t,\ y=\delta_t$ be that of $cx-4ay=2a+1$,

and $x=\gamma_t',\ y=\delta_t'$ that of $cx-4ay=2a-1$.

Then the general solution of (1.3) is given by the sets :

$$x=km+jl,$$
$$y=km-jl,$$
$$z=g_t,$$

where $j,\ k$ is a factor pair of h_t and $m,\ l$ is a factor pair of $2ah_t+1$;

$$x=km+jl,$$
$$y=km-jl,$$
$$z=g_t',$$

where $j,\ k$ is a factor pair of h_t' and $m,\ l$ is a factor pair of $2ah_t'-1$;

$$x=km+jl,$$
$$y=km-jl,$$
$$z=\gamma_t,$$

where $j,\ k$ is a factor pair of $2\delta_t+1$ and $l,\ m$ is a factor pair of $\{(2\delta_t+1)a+1\}/2$;

$$x=km+jl,$$
$$y=km-jl,$$
$$z=\gamma_t'.$$

where $j,\ k$ is a factor pair of $2\delta_t'+1$ and $l,\ m$ is a factor pair of $\{(2\delta_t'+1)a-1\}/2$.

(2) The following equations are solved by closely analogous processes.

(2.1) $x^2-b^2y^2=4bz(z+a)$ where a is any integer and b is prime.
Solution.

$$x=b\{jl+km\},$$
$$y=jl-km,$$
$$z=bjk \text{ or } bjk-a,$$

where j and k are any integers and $l,\ m$ is a factor pair of $bjk+a$ or $bjk-a$.

(2.11) $b(x^2-y^2)=4z(z+a)$, a and b as in (2.1).
Solution.

$$x=jl+km,$$
$$y=jl-km,$$
$$z=bjk \text{ or } bjk-a\ ;$$

j and k are any integers, and $l,\ m$ is a factor pair of $bjk+a$ or $bjk-a$.

(2.2) $4bx(x+y)+a^2=c^2z^2$, where a and c are any integers and b is prime.

Writing the equation in the form

$$(acz)^2=\{2bx(x+y)+a^2\}^2-\{2bx(x+y)\}^2,$$

the solution is found to be :

$$x=jl,$$
$$y=km-jl,$$
$$z=g_t,$$

where $x=g_t$, $y=h_t$ is the general solution of $cx-2by=\pm a$; j, k is a factor pair of h_t and l, m a factor pair of $bh_t\pm a$.

(2.3) $2b(x^2-y^2)+a^2=c^2z^2$, a, c any integers, a, b prime and not 2.

The solution is given by the sets of values :

$$x=jl+km,$$
$$y=jl-km,$$
$$z=g_t,$$

$x=g_t$, $y=h_t$ is the general solution of the linear equation $cx-4by=\pm a$; j, k is a factor pair of h_t and l, m a factor pair of $2h_t\pm a$.

The second set of values divides into two cases according as a is odd or even. If a is odd :

$$x=jl+km,$$
$$y=jl-km,$$
$$z=g_t,$$

where $x=g_t$, $y=h_t$ is the solution of $cx-4by=2b\mp a$; j, k is a factor pair of $2h_t+1$ and l, m is a factor pair of the integer $\{b(2h_t+1)\mp a\}/2$. If a is even :

$$x=jl+km,$$
$$y=jl-km,$$
$$z=g_t,$$

where $x=g_t$, $y=h_t$ is the solution of $cx-4by=\mp a$; j, k is a factor pair of $2h_t$ and l, m of $bh_t\mp a/2$.

R. L. G.

1198. . . . a " chart of progress ", a series of small squares by means of which a graph could be drawn to illustrate the scholar's ups and downs. Under this was the following : " At the end of the first week put a dot to represent position in form, at the end of the second week put another—join up with red ink. Continue this regularly each week. Unless you are head pupil, try to get a higher position each week ; KEEP ON TRYING. Rome was not built in a day."—Mark Grossek, *First Movement*, p. 86. [Per Mr. P. J. Harris.]

1199. For those who move in mathematical circles there is much virtue in a minus sign, but it is apt to bewilder the ordinary person who tries to think for himself.—Sir Napier Shaw, *The Drama of Weather*, p. 99.

[Per Mr. P. J. Harris.]

THE MECHANICAL INTEGRATION OF DIFFERENTIAL EQUATIONS.*

By D. R. Hartree, F.R.S., University of Manchester.

§ 1. Introduction.

In recent years there has been a great development in the design and use of machines for carrying out calculations of various kinds, and this is a technical advance of some importance, not only because of the saving of time and labour in calculations which would be carried out in any case, but still more because it makes it possible to carry out extensive calculations which would be too laborious and time-consuming to undertake without such mechanical or electrical aids.

Most of these calculating machines handle the processes of arithmetic (addition, subtraction, multiplication, division) singly or in combination, but a few have also been devised to handle calculations which involve the ideas and technique of the calculus, and in particular the idea of a rate of change of a changing quantity, which is a concept quite foreign to the purely arithmetical type of machine. One field for such machines, and one of particular practical importance, is the evaluation of the solutions of differential equations.

Differential equations which have no formal solution, or none convenient for numerical evaluation, are of common occurrence in a very wide range of applications of mathematics to problems both of pure and applied science, particularly in physics and electrical engineering, but also in other branches of engineering, in chemistry, biology, and possibly in economics as well ; and in the contexts in which they arise, it is often not only the qualitative form of the solution, but actual quantitative numerical values that are wanted. There are graphical and numerical methods for solving such equations, but graphical methods are limited in scope and accuracy, and are also tedious to carry out on a large scale ; and numerical methods, while ideally possible to carry out to any degree of accuracy, rapidly become more laborious and lengthy the more complicated the equation and the higher the accuracy required. Hence a mechanical means of evaluating the solutions of such equations, rapid, accurate enough for practical purposes, and easily applied to a wide range of equations, would be an important technical advance with a wide range of applications in pure and applied science.

Such an advance has been made by Dr. V. Bush of the Massachusetts Institute of Technology, by the development of a machine which he has termed the Differential Analyser (**6**), of which the first one was designed and built there. The name of this machine, by the way, seems to me scarcely appropriate, for the machine neither differentiates nor analyses, but, much more nearly, carries out the inverse of each of these operations ; however it is Dr. Bush's

* Based on a paper read at the Annual Meeting of the Mathematical Association, January 1938.

child, and I think he has the right to christen it. The original conception, in general terms, of such a machine, consisting of a number of integrating mechanisms connected together, is due to Kelvin (**1**), but the first practical design of a machine which could be made, and which would operate reliably and accurately when made, is due to Dr. Bush.

There are two or three other such machines now in operation in America ; in Europe there is one at the University of Manchester (**7**), which has now been in operation for three years and with which I have been chiefly concerned ; others are under construction for the Universities of Cambridge and of Oslo. The University of Manchester owes its machine to the generosity of its deputy treasurer, Sir Robert McDougall, who defrayed the cost of the machine to the University, to Dr. Bush for his advice and help in the design and building of it, and to Dr. Fleming, Director of Research at Metropolitan-Vickers Electrical Company, where the machine was built. The full-size machine is both large and costly, and there are already a number of small-scale versions of it in existence or under construction in this country, as will be mentioned later.

I want to divide this paper into three main sections. First, I want to give an idea of the general construction of the machine ; secondly, to indicate the way in which one has to think about a differential equation from the point of view of setting up the machine to evaluate its solution, and thirdly, to outline some actual applications.

§ 2. Construction. Integrators.

The machine consists of a number of units interconnected by shafts and gearing ; each unit is a translation into mechanical terms of an operation (integration, addition, etc.) which may be required in the solution of a differential equation ; each shaft represents some quantity occurring in the equation, and the rotation of the shaft represents (on a chosen scale) the magnitude of that quantity.

I will begin with the process of integration, which is the operation which the fundamental units of the machine have to carry out. Any continuously variable gear can act as an integrating mechanism. If (Fig. 1) one shaft (A) is arranged to drive a second shaft (B) at a

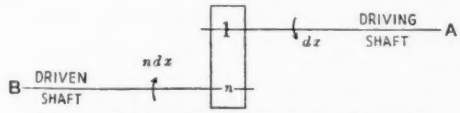


Fig. 1. Continuously variable gear as integrator.

gear ratio $n : 1$ (that is n turns of the driven shaft to one turn of the driving shaft), then for a rotation dx of the driving shaft when the gear ratio is n, the rotation of the driven shaft is $n\,dx$; if the gear ratio n is changing as the driving shaft rotates, the total rotation of

the driven shaft is the sum of the successive elements of rotation $n\,dx$ and so is $\int n\,dx$ in the aggregate.

For practical applications it is convenient to have a continuously variable gear for which there is no mechanical difficulty in the gear ratio passing through zero and becoming negative; and also, for accuracy of operation of the machine, the gear ratio must be capable of being set to a high degree of accuracy (of the order of 1 in 3000 or better if an overall accuracy of the machine of 1 in 1000 is contemplated). The principle of the form of variable gear used is shown in Fig. 2. A vertical wheel, whose plane is fixed, rests on a horizontal disc which can rotate about a vertical axis supported in bearings in a movable carriage, so that the distance y from the centre of the disc to the point of contact of the wheel with it can be varied. This distance corresponds to the gear ratio n, for if a is the radius of the vertical wheel, then if the disc rotates dx, the rotation of the wheel is $y\,dx/a$, provided there is no slipping (in the plane of the wheel). Hence if the displacement y is varying while the disc is rotating, the total rotation of the wheel is $\int y\,dx/a$. There is clearly no mechanical difficulty in making the gear ratio y/a pass through zero and become negative. It is interesting that a planimeter based on this principle * is much older than the Amsler type of planimeter which is now more familiar.

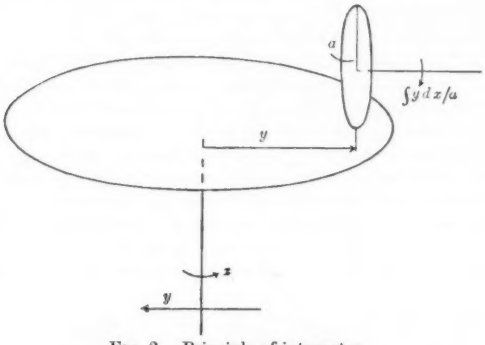


FIG. 2. Principle of integrator.

§ 3. APPLICATION TO A SIMPLE EQUATION.

To see how this integrating mechanism can be applied to the integration of differential equations, consider the simple equation

$$dz/dx = f(z),$$

* See reference (2), Vol. 15, p. 69c.

where $f(z)$ is a given function of z, possibly specified by an empirical graph or table and not necessarily by an analytical formula. Of course, by use of z as independent variable, the solution of this equation can be reduced to a quadrature, but I want to consider the equation as it stands with x as independent variable, as this will show better how the machine can be used. To see how it handles this equation, it is easiest to integrate the equation formally with respect to x, and to consider it in the form $z=\int f(z)\,dx$. This shows that, at each stage in the integration, the quantity $f(z)$ which must be supplied to an integrator as the integrand at that stage, must be related in a definite way to the value which the integral has reached at that stage. If this can be done (and I will show shortly how it can be done), and the disc of the integrator is driven by a motor whose rotation represents the quantity x, the rotation of the wheel of the integrator will be z. Thus the mechanical problem is to supply to the integrator a displacement $y=f(z)$ depending in a specified way on the value which the integral z has reached ; if this is done, the relation between the rotation of the disc (x) and the rotation of the wheel (z), will give the solution of the equation.

The way in which this is done is as follows (see Fig. 3). A graph

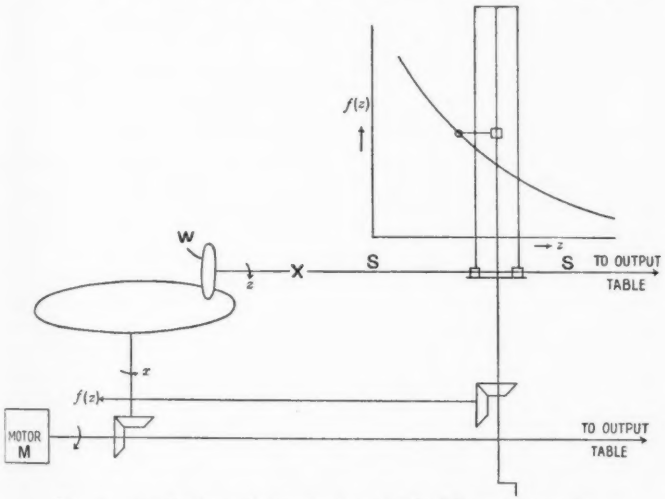


FIG. 3. Application of integrator to a simple differential equation

$$\frac{dz}{dx}=f(z), \quad \text{or} \quad z=\int f(z)\,dx.$$

of $f(z)$ against z, drawn beforehand, is placed on a table, called an input table, and a bridge, spanning the table in the direction of the

ordinates of the graph, is displaced in a direction perpendicular to itself by the rotation of the integrating wheel ; this is done by having a screw SS, constrained so as to be unable to move axially, and driven by the integrating wheel, passing through a nut fixed on the bridge. The bridge supports a carriage, which carries a pointer and which can be displaced along the bridge by the rotation of a handle. If the operator turns the handle so that the pointer follows the curve as the bridge is driven across the table by the screw, the total rotation of the handle at any stage (on a suitable scale and reckoned from a suitable zero) will be $f(z)$, where z is the total rotation of the screw at this stage from its zero position. If we now take a drive from this handle to displace the integrator, we will be supplying the integrator at each stage with a displacement $f(z)$ corresponding to the value which the integral z itself has reached at this stage, and this is just what we want to do in order to solve the equation. The rotation of the motor M, representing the independent variable x continually increasing, then drives the rest of the mechanism in accordance with the equation.

Finally, the solution (that is, z as a function of x) has to be recorded. This can be done graphically on a second table (output table) spanned by a similar movable bridge, supporting a similar carriage carrying a pencil instead of a pointer ; if the bridge is displaced in abscissa by the shaft whose rotation represents x, and the pencil holder is displaced along it by a shaft driven by the integrating wheel, the pencil will draw a graph of z against x.

§ 4. Torque Amplifier.

There is one important mechanical point to add. In a planimeter based on the integrator shown in Fig. 2, the integrating wheel has only to drive the counter which records the number of revolutions which it has made ; this is a light load and can easily be driven by the frictional force between the wheel and disc. In the evaluation of solutions of differential equations on the other hand, the integrating wheel (W, Fig. 3) may have to drive quite a lot of other mechanism ; in the example considered, it only has to displace the bridge on the input table, and the pencil holder on the bridge of the output table, but in less simple equations it may also have to displace or rotate not only one but several other integrators. This cannot be done by the frictional force between disc and wheel alone, and hence the shaft driven by the integrating wheel is broken and a mechanical relay or " torque amplifier " is introduced at X (Fig. 3), so that the portion of the shaft XSS rotates with the same velocity as the portion WX, but with a greatly increased torque.

The operation of the torque amplifier depends essentially on the principle of the capstan, familiar in elementary statics. The essential parts of a torque amplifier are shown diagrammatically in Fig. 4 ; the " input shaft " or driving shaft is the one to which the integrating wheel is fixed, and the " output shaft " or driven shaft is that to which the mechanism to be driven by the integrator is

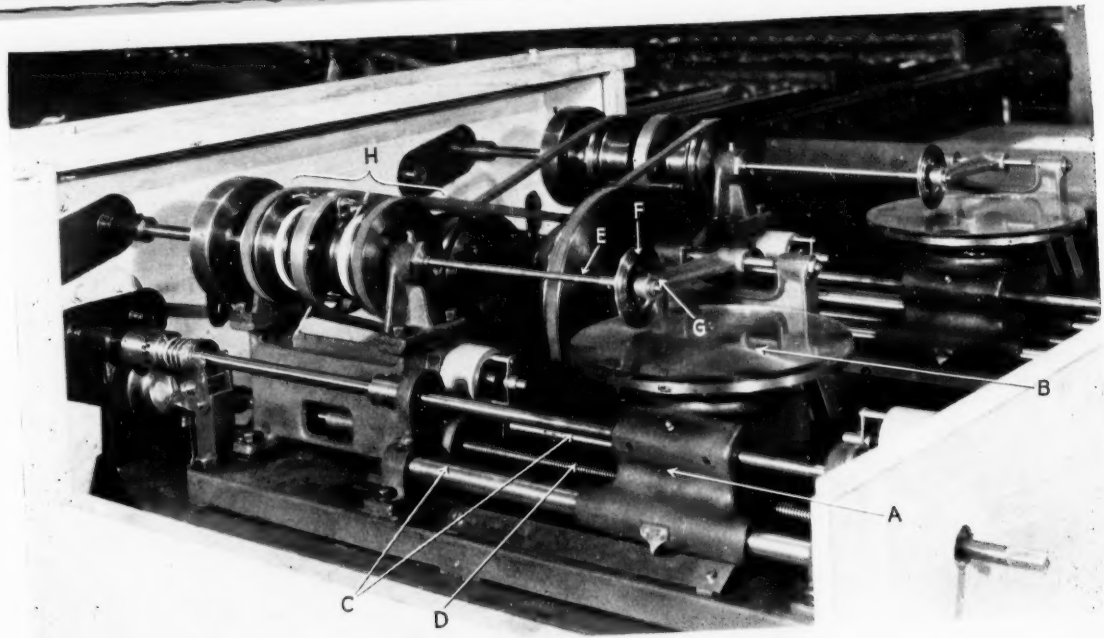


PLATE I. Integrator and torque amplifier.
(Photo by Metropolitan-Vickers Electrical Co., Ltd.)

PLATE II. General view of differential analyser at Manchester University.
(Photo by Metropolitan-Vickers Electrical Co., Ltd.)

connected. Drums are mounted coaxially with the shafts and are driven in opposite directions by a motor; they rotate independently of the shafts and are running continuously while the machine is in operation. Each shaft has an arm attached to it, and the only connexion between the two arms is through bands which pass round the drums. Now suppose the input shaft rotates so that the arm *A* in Fig. 4 moves down into the paper; the band on the right-hand drum tightens and the friction between it and the drum is increased; the consequent additional tension in the band builds up exponentially along the band (if it is perfectly flexible), so that the couple on the arm *B* fixed to the output shaft is greater than that on the arm *A* fixed to the input shaft; a torque ratio of 60 or 80 to 1 is quite easily attainable, and by using two of these mechanisms in series, taking the output shaft of the first as the input of the second, a torque ratio of the order 8000 to 1 can be obtained.

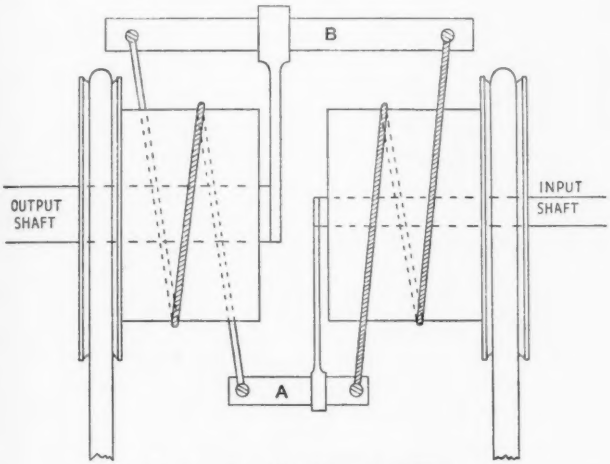


FIG. 4. Principle of torque amplifier.

Plate I shows an integrator and its associated torque amplifier. The carriage (*A*), carrying the integrating disc (*B*), slides on guide rods (*C*) and is displaced by the rotation of a screw (*D*) passing through a nut which forms part of the integrator carriage. One end of the shaft (*E*), carrying the integrating wheel (*F*), is supported in a fixed bearing (*G*) and the other end is carried in a hole drilled in the output shaft which is rotating at the same speed, so that at that end there is no friction. The torque amplifier (*H*) can be seen behind the integrator.

PLATE II. General view of differential analyser at Manchester University. (Photo by Metropolitan-Vickers Electrical Co., Ltd.)

§ 5. Construction. Remainder of Machine.

Plate II shows a general view of the machine at Manchester University. On the near side of the machine in the picture there are eight integrators (*A*) in pairs in four cases, and the output table (*B*) on which a graph of the solution of the equation can be drawn by the machine. On the far side there are six input tables (*C*) for supplying to the machine information about functions occurring in the equation from previously drawn graphs of those functions ; one of these (*D*) is a special table whose object will be mentioned later.

Down the centre is an arrangement of shafts and gearing for connecting together the different units of the machine so as to handle different equations. There are two sets of shafts at right-angles, one set running parallel to the length of the machine and one set across it ; these sets of shafts are in different planes so that any shaft of one set can be connected to any one or more of the other set by helical gears. The cross shafts are those which directly drive or are driven by individual units of the machine, and the longitudinal shafts are used to interconnect the various units.

A graphical record of the solution is most convenient if a survey of its general behaviour is required, but is not convenient for permanent record, or if accurate numerical values of the solution are wanted. An alternative method of recording is provided by a set of revolution counters (*E*) driven by the machine, and a camera (*F*), driven by the shaft whose rotation represents the independent variable, which takes photographs of the counters, so giving directly a table of numerical values of the solution at equal steps of the independent variable.

Close-up views of some of these units and of some mechanical details are shown in Plates III and IV. Plate III shows an input table ; the bridge (*A*) is displaced, in a direction perpendicular to its length, by the operation of the machine ; the carriage (*B*) carrying the pointer (*C*) is displaced along the bridge by an operator turning the handle (*D*) ; the rotation of this handle also turns a cross shaft (*E*) from which a drive can be taken to the point in the machine to which the information contained in the graph is to be supplied. Experiments are now in progress on an automatic curve follower, operated by a photoelectric cell, to take the place of an operator.

The output table is similar and is not illustrated. It consists of a pair of similar bridges connected together, each carrying a pencil holder, so that graphs of two quantities occurring in the equation (for example the dependent variable and its first derivative) can be drawn by the machine.

The special input table, which again is similar, has two independent bridges, one carrying a pencil holder and the other a pointer. The pencil draws a curve which can subsequently be followed by the pointer, so that the information expressed by the portion of the curve drawn by the pencil at one stage in the process of solution, can be fed back to the machine at a later stage. This unit is an addition

PLATE III. Input table.
(Photo by Metropolitan-Vickers Electrical Co., Ltd.)

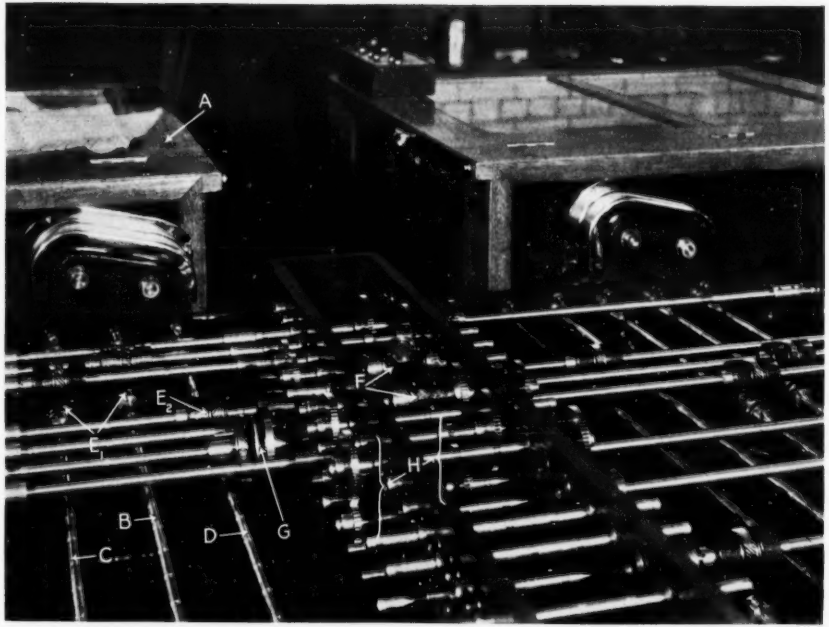


PLATE IV. Mechanical details of interconnection of units of differential analyser.
(Photo by Metropolitan-Vickers Electrical Co., Ltd.)

to those of Dr. Bush's original machine, and was first designed and built for the machine at Manchester. It is used in the solution of the equations of a system with a time-lag, in which (for example) dy/dt at time t depends not only on the value of y (and possibly of other quantities) at that time t, but on the value of y and possibly other quantities at a previous time $t - T$, where the time-lag T may be constant or variable. Such equations have not been much studied formally, but they have arisen in connexion with two quite separate technical problems (see references **16**, **20**).

Plate IV shows some of the mechanical details involved in the interconnexion of different parts of the machine. It shows the back of a case (A) containing a pair of integrators, the two cross shafts (B, C) driving one of the integrators, one shaft (B) displacing and the other (C) rotating it, and the shaft (D) driven by it. Each of these shafts has one wheel of a helical gear pair (E) fixed to it, the other wheel of the gear pair being on one or other of the longitudinal shafts. The helical gears are of two kinds, right-handed (for example, E_1) and left-handed (E_2); the replacement of a right-handed by a left-handed gear reverses the direction of rotation of the longitudinal shaft relative to the cross shaft, and in setting the machine for an equation, the signs are arranged by the use of right or left-handed gears at the different connexions between longitudinal and cross shafts.

Plate IV also shows two adding units (F), which are mechanisms for interconnecting three shafts so that the rotation of one is the sum of the rotations of the other two; the arrangement of gears in each of these is similar to that in the differential of a motor-car. This plate also shows a mechanism (G) termed a front-lash unit, used for compensating the backlash in a drive if it is appreciable; it consists of an epicyclic gear train so arranged that the shaft driven by the unit turns 10 per cent. faster than the shaft driving it; this occurs for a fraction of a revolution only, and this fraction is adjusted by trial and error so that the consequent advance of the driven shaft relative to the driving shaft just compensates the backlash. The plate also shows a train (H) of spur gears for arranging the scales on which different shafts represent the variables in the equation.

The way in which the machine is constructed, largely of standard parts (gear wheels, axles, etc.) connected together, is rather reminiscent of the Meccano system, and my first impression on seeing the photographs of Dr. Bush's machine was that they looked as if someone had been enjoying himself with an extra large Meccano set. This suggested the possibility of trying to build, wholly or at least mainly out of standard Meccano parts, a model version of the machine which would serve as a demonstration model to illustrate qualitatively the general principles on which it works. The result was successful beyond expectation; we found it possible to build, almost entirely in Meccano, a model differential analyser (**10**), which would work not only qualitatively but quantitatively to an accuracy

PLATE IV. Mechanical details of interconnection of units of differential analyser.
(Photo by Metropolitan-Vickers Electrical Co., Ltd.)

of about 2 per cent., which could be (and has been) improved by more attention to details of design. Some original work (**16**) has been done on this model, and it has also been valuable for trying out ideas for new units of the machine, such as the special input table and an automatic curve follower, in an experimental form before incorporating them in the full-size machine. This model is shown in Plate V; a fourth integrator has been added subsequently. I regard the success of this model as a striking tribute to the versatility and practical value of the Meccano system.

Other small-scale machines, either built of standard parts or workshop built, have been constructed at Cambridge by Professor Lennard-Jones and at Belfast by Dr. Massey (**11**), and several others are under construction or contemplated. The construction of a somewhat smaller and simpler version is not beyond the resources of schools, and some readers may be interested to know that such a model was built a few years ago at Macclesfield Grammar School by Mr. R. Stone and some of his VI Form boys.

§ 6. Use and Operation of the Machine.

I want now to give some idea of the way in which one has to think about a differential equation in order to see how to set it on the machine.

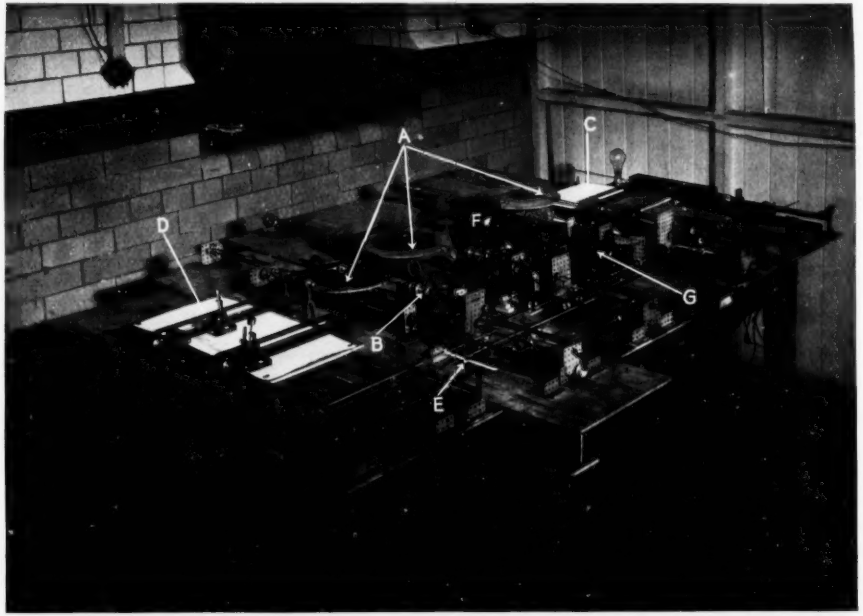
The machine set-up is first worked out on paper, and the actual setting up of the machine by fitting the gear wheels, etc. in the appropriate places is done from the final diagram which is the result of the paper work. In drawing out the set-up diagram a notation is used in which each unit is represented by a symbol indicating in a simple form the mechanical operation performed by the unit; the notation introduced by Bush (**6**), with one or two additions and alterations which our experience has shown to be convenient, is shown in Fig. 5 (p. 351).

The process of working out a machine set-up consists of three stages, as follows:

(i) The general scheme of the kind of units to use and the way to connect them together is drawn out without reference to scales or directions of rotation of the various shafts.

(ii) Scales on which the rotations of the different shafts shall represent the variables in the equation, convenient from the point of view of time taken for solution, the limits on the displacements of integrators, etc., are chosen; this includes fixing the ratios required for gear trains between different shafts.

(iii) The actual units of the machine, and the location of the different shafts, gears, etc., to be used, are drawn out on a skeleton plan of the machine; the positive directions of rotation of the various shafts are settled, and the sign of the helical gears (right-hand or left-hand) at each connexion between a longitudinal shaft and a cross shaft is determined.



A Integrators
B Torque amplifier
C Input Table
D Output Table
E Longitudinal shafts
F Motor driving torque amplifier
G Motor driving independent variable shaft

PLATE V. Model differential analyser at Manchester University, mainly constructed from standard Meccano parts.
(Photo by Guttenberg.)

An idea of the general nature of the process can be shown by a simple example, and for this purpose I will consider the equation

$$\frac{d^2y}{dx^2} = -y. \quad \text{.....(1)}$$

Fig. 6 shows the final result of stage (i) ; it is obtained as follows.

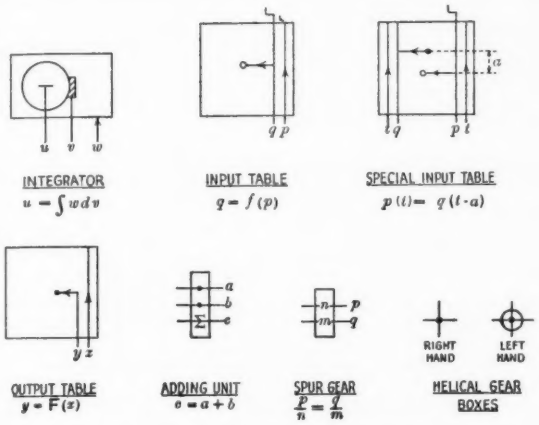


FIG. 5. Notation for units of machine in set-up diagram.

The letters p, q, r ... stand for the quantities whose values are measured by the rotations of the respective shafts. Under the symbol for each unit is given the formula showing the relation between these rotations which results from the operation of the unit.

On the input and output tables,

an open circle ○ represents a pointer for following a curve;

a black circle ● represents a pencil for drawing a curve.

The longitudinal shaft nearest the integrators is always assigned to the independent variable, here x ; that is to say, the rotation of this shaft represents x, on a scale to be settled in stage (ii). Let the rotation of the next shaft in the diagram represent d^2y/dx^2 (again on a scale to be fixed later). Then since integrator I is supplied with d^2y/dx^2 as integrand and x as variable of integration, the rotation of the shaft carrying the integrating wheel will represent dy/dx ; this shaft is connected to the third longitudinal shaft in the diagram. A drive is then taken from this longitudinal shaft to displace integrator II, and a drive from the x shaft rotates this integrator, so that it evaluates $\int (dy/dx)\,dx = y$. Hence the rotation of the shaft carrying the integrating wheel of integrator II represents y. But we require d^2y/dx^2 to be equal to y (or rather to $-y$, but signs are irrelevant at

this stage), and we began by taking the rotation of the middle longitudinal shaft to represent d^2y/dx^2 ; thus we can obtain the relation we require by making this shaft be driven by the integrating wheel of integrator II. If this is done, the relations between the rotations of the different shafts must satisfy the equation (Fig. 1) (apart from mechanical failure on the part of the machine), and the rotation of the motor driving the x shaft will drive the rest of the machine in accordance with the equation. It should be noticed in passing that the relations between the motions of the different parts are purely kinematic relations, and the operation of the machine does not require the x shaft to have any definite (or even a uniform) speed ; also that in this case the process of solution is purely automatic, no input table requiring an operator is needed.

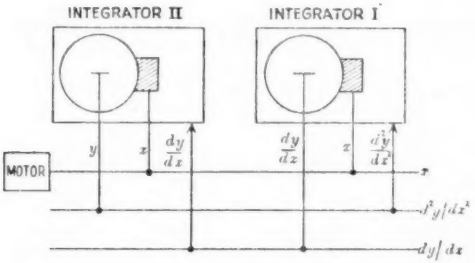


FIG. 6. Set-up diagram for equation $d^2y/dx^2 = -y$. First stage ; scale factor and signs disregarded.

The final result of stage (ii) for this equation is shown in Fig. 7.

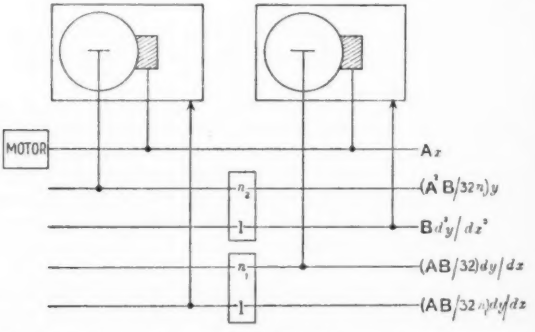


FIG. 7. Set-up diagram for equation $d^2y/dx^2 = -y$. Second stage ; scale factors and gear ratios introduced.

Here scale factors A, B, ... are introduced to give the number of

shaft turns to be used to represent unit change of the quantity measured by the rotation of each shaft. The scale factors of the integrators, which relate the numbers of turns of the two driving and one driven shafts (and which depend on the radius of the integrating wheel, pitch of screw displacing the integrator, etc.), are introduced, and also ratios of gear trains between each longitudinal shaft driven by one integrator and the shaft, driven by it, which displaces the other integrator. Let the rotation of the x shaft be Ax turns, and the rotation of the d^2y/dx^2 shaft be Bd^2y/dx^2 turns; then the rotation of the integrating wheel of integrator I will be

$$(AB/32)\int(d^2y/dx^2)dx = (AB/32)dy/dx \text{ turns};$$

this rotation is geared down in the ratio n_1 to 1 to displace integrator II, so that the displacement of this integrator is $(AB/32n_1)dy/dx$ turns, and the rotation of its integrating wheel is similarly $(A^2B/32^2n_1)y$ turns. If this is geared down n_2 to 1 to drive the d^2y/dx^2 shaft, the rotation of this shaft will be $(A^2B/32^2n_1n_2)y$ turns. But we started by taking the rotation of this shaft to be Bd^2y/dx^2 turns; hence in order that the rotations of the shafts, measured on these scales, should satisfy equation (1), we must have

$$A^2B/32^2n_1n_2 = B \quad \text{or} \quad A^2 = 32^2n_1n_2; \quad \ldots\ldots\ldots\ldots\ldots(2)$$

the disappearance of the scale factor B is a consequence of the fact that the simple equation I have considered is linear in y.

In this particular example, this is the only relation which has to be satisfied by the scale factors and gear ratios. Usually there will be several such relations, although fewer than the number of these parameters, and one has a good deal of liberty in adjusting the values of the parameters, consistently with the relations to be satisfied, in order to obtain the most convenient values for practical working.

The value of A, and the speed of the x shaft, determine the time required for a solution; for example, if the speed of the x shaft is 300 r.p.m., and the range of x to be covered is from $x=0$ to 30, the time required for a solution is $A/10$ minutes. If this time is to be about half an hour, a convenient value of A is 256; then the condition (2) is satisfied by $n_1=n_2=8$.

Stage (iii) can now be undertaken with the result shown in Fig. 8; the gear ratios $n_1=n_2=8$ have to be obtained by a train of gears, since the small wheel of a single pair giving an 8 : 1 ratio would be too small to fit the shafts. Each shaft is marked with the quantity whose value is measured by its rotation, and the positive senses of rotation are marked by arrows showing the direction of movement of the top of each shaft for increase of that quantity; the relation between these directions for the driving and driven shafts of each unit depend on the details of its construction. Right-handed or left-handed helical gears connecting longitudinal and cross-shafts

are finally marked according to the relation required between the positive directions of the two shafts ; the notation has already been shown in Fig. 5.

The connections to the output table are shown arranged so that the solution is recorded in the form of a graph of dy/dx against y (not against x). This is a convenient test of the mechanical accuracy of the machine, as the curve described should be a circle, and it is easy to see how the recorded solution differs from this.

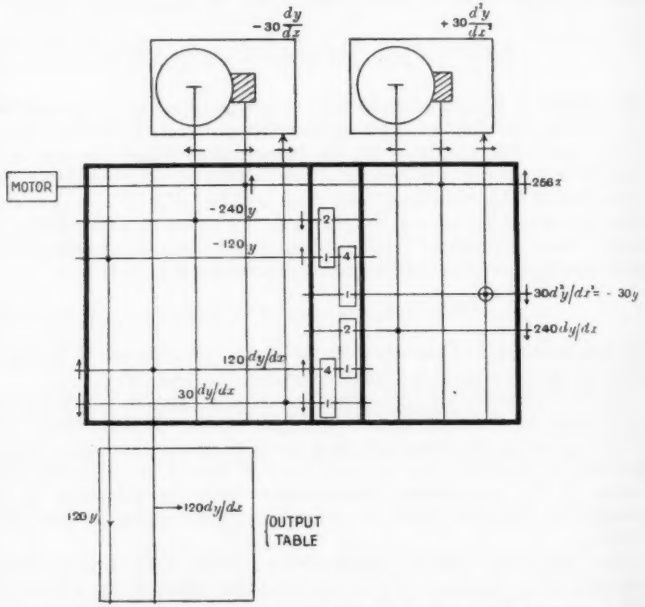


FIG. 8. Set-up diagram for equation $d^2y/dx^2 = -y$. Third stage: detailed plan of layout on machine, including determination of signs. The heavy lines show the framework of the machine.

Initial conditions for the solution of the equation are supplied to the machine in the form of initial displacements of the integrators; when this has been done, the rotation of the x shaft drives the rest of the machine so as to evaluate the solution of this equation from these initial conditions.

I have only considered a very simple example, as a more elaborate one would have taken too much space to discuss fully, but it may be of interest if I mention the few general points which often arise in dealing with less simple equations.

One much-used way of interconnecting two integrators is shown in Fig. 9. In Fig. 6 the rotation of the integrating wheel of integrator I is made to *displace* integrator II, that is to say the result

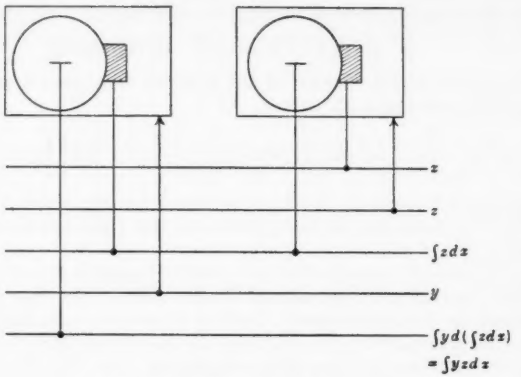


FIG. 9. Integration of a product.

of one integration is taken as the integrand of another, so that what is finally evaluated is a second integral. In Fig. 9 the rotation of the integrating wheel of integrator I is made to *rotate* integrator II, that is to say the result of one integration forms the variable of integration for the second, and by this means the integral of a product can be evaluated.

It is interesting and very convenient that the integral of a product can be evaluated without the product itself being formed, for the integral of the product is often required as one step in working out a machine set-up, whereas a product (as apart from its integral) is comparatively seldom required and is difficult to construct if it is required. To form a product it is necessary to have a mechanism which will make one shaft rotate through an angle which is the product of the rotations of two other shafts, and a simple, accurate, and automatic mechanism for doing this, which will enable either factor to go through zero (so that the use of logarithms is impracticable), is surprisingly difficult to devise ; perhaps the best method, though it may seem a very sophisticated one, is to use two integrators and to evaluate $yz = \int y\,dz + \int z\,dy$, but it may be difficult to spare two integrators for this purpose. However in the three years for which the differential analyser at Manchester University has been working, we have never needed to form such a product on the machine, and this may be taken as an indication of the rarity of this particular situation.

Very often in setting an equation on the machine it is most con-

venient to look at it in the form it takes after formal integration once with respect to the independent variable, just as was done in § 2 of this paper. This often shows that for some of the integrations it is most convenient to use other variables of integration than the independent variable. For example, the equation

$$y''' = -yy'' + \beta[y'^2 - 1] \quad (\beta \text{ constant})$$

which occurs in the theory of the laminar boundary layer in fluid motion (cf. reference **17**), gives

$$y'' = -\int yy'\,dx + \beta\left[\int y'^2\,dx - x\right] = -\int y\,dy' + \beta\left[\int y'\,dy - x\right];$$

y' and y are formed as the first and second integrals of y'', and can be used as variables of integration on the right-hand side in the construction of y''.

If a quadratic, exponential, or circular function of any one of the variables occurs in the equation, this function may either be fed into the machine by an operator following a curve on an input table, or it may be generated by the machine by the evaluation of an integral or by the solution of an auxiliary equation ; for example :

$$y^2 = 2\int y\,dy\,; \quad z = e^w \text{ satisfies } \frac{dz}{dw} = z.$$

The same principle can be used, though not quite so simply, to generate other functions such as reciprocals, logarithms, square roots, etc. Each such use of one or more integrators dispenses with the need for a operator, and makes the process of the solution so much more nearly automatic, and the main reason for the large number of integrators in most such machines is in order to have plenty available for subsidiary uses such as these.

The preliminary set-up diagram (corresponding to Fig. 6 for equation (1)) for the equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + n^2x(1+ax) = f\cos pt \quad \text{.....(3)}$$

is shown in Fig. 10 to illustrate some of these points, and as an example of a machine set-up for an equation more complicated than that considered in detail above but still comparatively simple. Here integrators IV and V generate $f\cos pt$ by integrating the auxiliary equation $d^2z/dt^2 = -p^2z$, using the connexions shown in Fig. 6, the amplitude and phase of this oscillation being controlled by initial settings of the integrators ; integrator III constructs $x(1+ax)$ as $\int(1+\frac{1}{2}ax)dx$ (the term 1 in the integrand being supplied by the integrator displacement at $x=0$), and integrators I and II integrate the derivatives occurring explicitly in the equation itself.

The time required to work out the machine set-up diagram for any given equation, and actually to set the machine up from the diagram,

varies from a few hours to a day or two depending on the simplicity or otherwise of the equation and its likeness to other equations previously met with; it is here that practical experience of the machine counts for a lot, as often there are alternative ways of setting up an equation, and the only basis for judging between them, and for fixing scales, etc., is previous experience with similar situations.

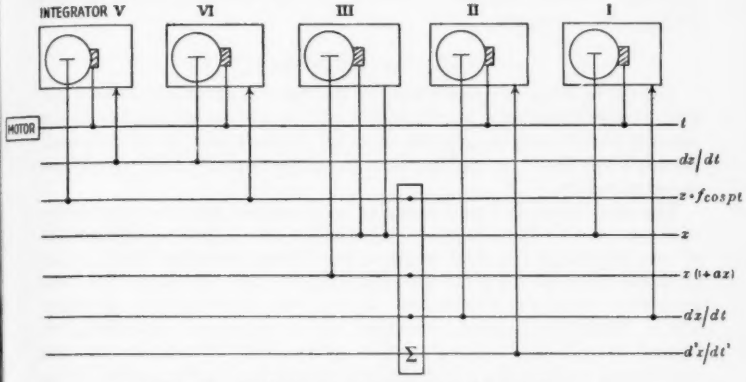


FIG. 10. Preliminary set-up diagram for equation
$\ddot{x}+k\dot{x}+n^2x(1+ax)=f\cos pt.$
(Scale factors and signs disregarded.)

Once the machine is set up, the running of a solution may take from a quarter of an hour up to an hour or more, depending on the range over which a solution is wanted, and features such as the possible need to change scales in the course of the run in order to retain adequate accuracy throughout and still to prevent integrators overrunning their permissible range of movement.

Thus if only one solution of an equation is required, the saving of time by the use of the machine may not be very large. But if a number of solutions is required, the same machine set-up may remain in use for a week or a month or more; then the time required to set up the machine becomes a small fraction of the whole time spent on the problem, and this gives the most economical conditions for the use of the machine. Thus the machine is most useful when a large number of solutions of the same equation, or of essentially similar equations differing perhaps in numerical values of parameters or in functions supplied from input curves, is required. This is often the case in practical problems, when a set of solutions for different values of initial conditions, or of a parameter, is wanted, or when solutions are required satisfying boundary conditions specified at both ends of the range of integration. In the latter case, where numerical or mechanical methods of integration are used, it is

necessary to start from one end of the range, and to adjust the boundary values of certain derivatives at this end until the solution satisfies the boundary conditions at the far end. Unless the equation is linear, this can only be done by trial and error, so that a number, and perhaps quite a large number, of solutions have to be evaluated before a useful one satisfying the specified conditions at the two ends is found. Thus the differential analyser is particularly useful in this kind of "two-boundary" problem, which is of rather common occurrence in practice.

§ 7. Applications.

A large number of applications of the various differential analysers now in operation have been made, and it would be impossible to say enough about each to explain them all in the space at my disposal. Hence I will take two and consider them particularly, and give references to others.

The first example I will consider is concerned with the possibility of the presence of sub-harmonics in the forced oscillations of a system of one degree of freedom under a non-linear restoring force; this problem came to me from an authority on loud speakers, who was concerned with the occurrence of such sub-harmonics in practice. For a vibrating system described by a non-linear equation, we cannot describe the motion as the superposition of independent free and forced oscillations; but under a simple harmonic applied force, there may be a possible periodic motion which corresponds to the forced oscillation alone in the linear case, and this is the motion which I mean by the forced oscillation. This motion will not be simple harmonic (apart from exceptional cases) but will contain higher harmonics of the applied frequency; the question is whether it is possible for it to contain components whose frequencies are sub-multiples of the applied frequency. The system chosen was one with damping and a quadratic departure from a linear restoring force, so that its equation was (3) above.

From the point of view of using the machine on this equation there were two main problems, namely how best to obtain a solution corresponding to a forced oscillation, and how best to identify the sub-harmonic in the solution if there was one. To obtain the forced oscillation it is necessary either to try different sets of initial conditions until finding those which give a periodic solution, or to take a single solution from arbitrary initial conditions through many periods of the applied force, in the hope that in the course of time the effect of initial conditions will be damped out, as is the case for a linear system, so that only the forced oscillation is left. With either process, and by whatever way the integration is done, a considerable amount of preliminary work is required for each useful solution evaluated; but since all solutions use the same machine set-up, the conditions are the most favourable for the use of the machine, as explained in the last paragraph of the previous section, and using the machine the possible occurrence of a sub-harmonic over quite

a range of values of the parameters can be investigated in the course of a few days.

The sub-harmonic, if one is present, is most easily identified by drawing the graph of the solution of the equation in the form of a curve, not of x against t, but of x against cos pt or sin pt, so producing a diagram similar to those termed "Lissajous figures" in acoustics. If x is periodic with the same frequency as the applied force, then as sin pt goes through a cycle, so does x, so that the curve of x

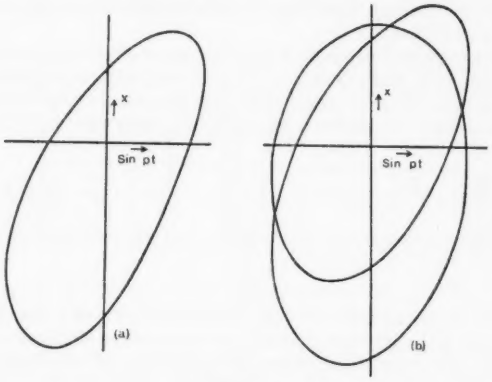


FIG. 11.

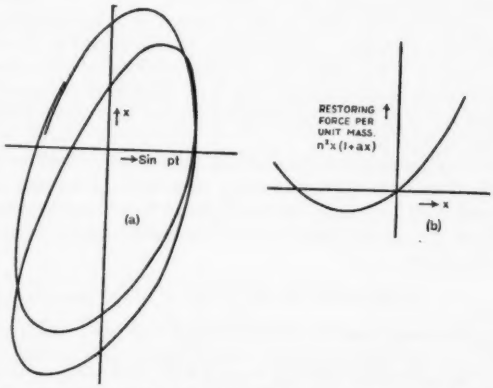


FIG. 12. Machine solution showing occurrence of subharmonic in forced oscillation of system with non-linear restoring force.

against sin pt is a single closed loop (Fig. 11*a*). But if a sub-harmonic of half frequency is present, then as sin pt goes through a

cycle, the sub-harmonic only goes through half a cycle ; sin pt is to go through two cycles before the sub-harmonic repeats, so that the curve of x against sin pt is a double loop (Fig. 11b).

Fig. 12a is a reproduction of a machine solution showing a close approach to a curve joining up in a double loop, and closer approximations have been obtained, showing the possibility of occurrence of the sub-harmonic, although, for this equation, it only occurs for amplitudes of oscillation greater than those corresponding to loud speaker practice ; there is no sign of a sub-harmonic until the amplitude of the oscillation goes beyond the minimum of the restoring force curve in Fig. 12b.

The other application which I shall consider shortly * is one which is remarkable both for the character of the equations and the nature of the solutions. This is concerned with electrical surges on a finite transmission line. Consider a line (Fig. 13) into one end of which a

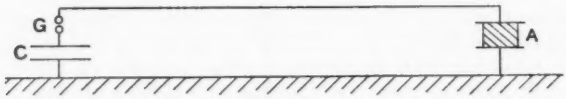


FIG. 13.

condenser C is discharged through the breakdown of a spark gap G, the other end of the line being fitted with a resistive element A (that is to say, one for which the potential difference V depends on the current i, not on its derivative or integral) with a relation between V and i, which is not necessarily linear, say $i = G(V)$ which may be a purely empirical function of its argument. We were particularly concerned with cases in which this element is a lightning arrester constructed from material for which i is proportional to the cube or fourth power of V. Further let c be the velocity of propagation of an electrical disturbance along the line, which is assumed to be distortionless (i.e. c is assumed independent of the wave form), and cT the length of the line.

The electrical disturbance on the line can be expressed as the sum of two waves, one travelling in each direction along the line ; if $f'(t-x/c)$ and $F'(t+x/c)$ are the voltages in these two waves, it can be shown that the relation between current and voltage at the right-hand end ($x=0$) is

$$f'(\xi) - F'(\xi) = ZG[f'(\xi) + F'(\xi)], \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

and that at the condenser at the left-hand end ($x = -cT$),

$$t_1[f'(\xi+2T) + F'(\xi)] = -f(\xi+2T) + F(\xi), \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where Z and t_1 are constants depending on the characteristics of the line and the capacity of the condenser C.

The form of these equations is unusual for two reasons, first

* For a fuller account, see reference 20.

because of the implicit form of the equation (4) relating f' and F', and secondly because while this equation relates the unknown functions f and F at the same value of the argument, equation (5) relates them at different values, on account of the finite velocity of travel of an electrical disturbance along the line. The latter situation is that with which the special input table of the machine was designed to deal, and using it, it is possible to evaluate the solution of these equations despite their unconventional form.

Fig. 14, showing the time variation of the potential across the element A, is reproduced from the machine graph of a solution, and

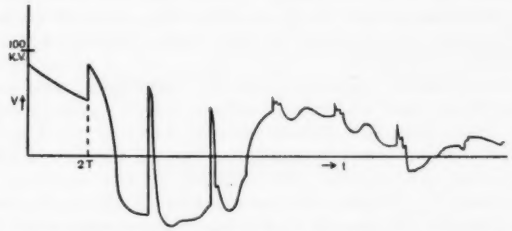


FIG. 14. Example of machine solution giving time variation of voltage across lightning arrester A (see Fig. 13).

is rather startling at first sight; the discontinuities arise as the results of the successive reflexions of the discontinuity arising from the breakdown of the spark gap G in Fig. 13, and are to be expected; the several wriggles in the otherwise fairly smooth portions of the curve between the discontinuities look surprising and might at first suggest careless following of an input curve by an operator; but this behaviour of the solution is repeatable in detail, and each one of the wriggles can be explained by a detailed discussion of the potentials and currents. As further evidence that these characteristics of the solution are real, Fig. 15 is a reproduction of an oscillogram of the potential difference across a lightning arrester at the

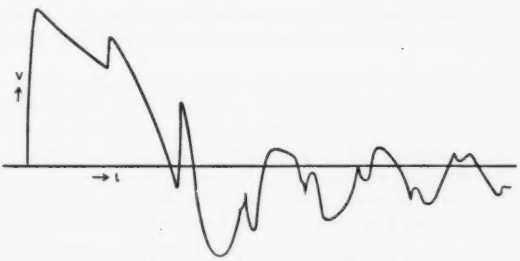


FIG. 15. Example of oscillograph of voltage across lightning arrester on an actual finite transmission line.

end of an actual finite line ; * only qualitative agreement with Fig. 14 can be expected, as the solution shown in Fig. 14 was worked before the experiments were done, and the characteristics of the actual line and arrester differ from those for which Fig. 14 was evaluated ; but there is general agreement between the kind of behaviour of the potential in the two figures. The form of the solution shown in Fig. 14 shows the forbidding nature of these equations for numerical solution ; the evaluation of such a solution would hardly be practicable without mechanical aid such as is provided by a differential analyser.

These are only two out of many applications that have been made, a list of published papers on these and other applications has been included in the references at the end of the present paper, in case they would be of interest to any reader.

The main field of application of the differential analyser is to equations which have no formal solution, but occasionally equations arise for which there is a formal solution, but one which is very inconvenient for numerical evaluation. Then if numerical values for a special case are required, the solution of the equation by the machine may be quicker than evaluation of the formal solution. This may occur, for example, with linear equations of the third or higher order with constant coefficients, when the evaluation of a solution may involve finding the roots, some of which may be complex, of an equation of third or higher degree, and then calculating and summing the corresponding exponential or damped harmonic terms, which is a very tedious business.

It must be realised, of course, that the differential analyser can only evaluate the solution of an equation with definite numerical values of all coefficients occurring in it, from definite numerical initial conditions ; it cannot give a general solution, even if a formal solution does exist. For this reason it is not likely to be of much interest to the pure mathematician engaged on the general theory of differential equations, and for the same reason it is not to be regarded as a substitute for any of the formal study of differential equations ; it is only a last resort when this study fails to give the kind of results required. But in practical problems it so often happens that this last resort is the only practical means available, and it is in such contexts that the machine is likely to be most valuable.

I might add that though the machine is only applicable directly to *ordinary* differential equations, a method has been devised for using it for the approximate integration of some types of *partial* differential equations (**19**) ; two applications of this method have been worked through successfully, but neither has yet been published.

In conclusion I should explain that in dealing so fully with the differential analyser I do not want to imply that it is either the only

* I wish to thank the Director, National Physical Laboratory, for permission to reproduce this figure, which is copied from an oscillogram taken in the High-Tension Department of the Laboratory.

instrument for carrying out the solution of differential equations, or that it is the last word in the subject. For particular kinds of equations other instruments are available, and accounts of some of these are included in the references (3), (5); but these instruments are usually designed to deal each with a special type of equation, and they are limited to these particular types, whereas an important feature of the differential analyser is the very wide range of equations which can be handled by different methods of interconnecting the same elements of mechanism. Also it may be possible to devise electrical methods of doing all that the differential analyser does mechanically, and perhaps more; but other machines of similar scope are only in the experimental stage, if as far advanced as that, and at present the differential analyser seems to me the most interesting as having by far the widest scope of any machine yet actually constructed and proved by extensive practical experience.

ACKNOWLEDGMENT.

I AM indebted to the Institution of Electrical Engineers for permission to reproduce here Plates I and II, and Figs. 5, 13, 14, 15, which have been used to illustrate papers now in course of publication in the Journal of the Institution.

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See also (**12**) for references to various applications in electrical engineering.

1200. A Kenya headmaster sends a story which illustrates the zig-zag course that education may take in aboriginal Africa:

On going into a class the other day to see how a young African teacher was progressing, I found the following problems on the board:

If a hen has twenty chickens in one week, how many will she have in three weeks?

If a man and his wife have thirty-eight children, how many have they each?

When asked why he gave such unpractical sums, he was quite offended. "These do not represent the *truth*, sir," he said; "they are only arithmetical examples!"

One is not quite sure that examples of the hypothetical method could not be found nearer home.—From "At Random" in the *Observer*, 5th December, 1937. [Per Dr. W. G. Bickley.]

A DIFFICULT CONVERSE.

By J. P. McCarthy.

If the bisectors of the base angles of a triangle are equal, the triangle is isosceles.

Several solutions of the problem are given in the *Gazette* for July 1932, and again in the issue of May 1933. It is stated in the former that the problem was mentioned in the *Lady's and Gentleman's Diary* of 1859 and of 1860. The history of the problem is interesting. It appears that it was first proposed in 1840 by Professor Lehmus of Berlin to Jacob Steiner, who had been Professor of Geometry at Berlin since 1834 and held that post until his death in 1863.

In a paper, Crelle's *Journal* (Bd. 28, 1844),* Steiner says that he solved the problem at once and communicated it immediately to Lehmus, but as in the meantime several solutions had appeared in different publications, not so elementary as he and Lehmus desired, and as the problem had been erroneously attributed to himself, he decided to publish the solution he had found at the time, which was considered by Sturm, who had seen many solutions, to be the most elementary. The following is Steiner's solution ; the figures are his and the translation is free.

Given that $\alpha = \alpha_1$, $\beta = \beta_1$, and $AD = BE$ or $a = b$, assume that $\alpha > \beta$. Then in the triangles ADB, BEA we have AB common, $AD = BE$, $\alpha > \beta$. Hence

$$BD > AE \quad \text{or} \quad d > e,$$

and $\angle ADB > \angle BEA$ because $\alpha_1 + \alpha + \beta > \beta_1 + \beta + \alpha$.

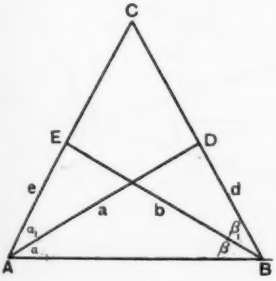


Fig. 1.

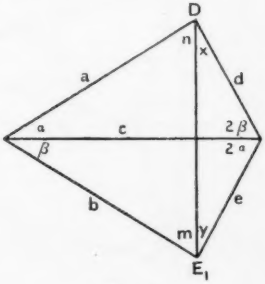


Fig. 2.

Place the triangles as in Fig. 2 so that they stand on opposite sides of c $(= AB)$ and where the sides are equal to those denoted by the same letters in Fig. 1. Now $a = b$: if we draw DE_1, then $n = m$, and since in Fig. 1, $D > E$ or $\angle ADB > \angle BEA$,

* See also *Gesammelte Werke*, II, 1882.

then $$x > y;$$

and so $$e > d,$$

which contradicts the previous statement that $d > e$. Thus α and β cannot be unequal, and hence ACB is isosceles.

Steiner says that this is the first solution he obtained.

He then considers the cases that arise if the external angles are bisected by lines meeting the opposite sides in D and E, the lines being denoted by a_1 and b_1. There are, he says, three cases:

either (1) $a = b$,

(2) $a_1 = b_1$,

or (3) $a = b_1$ or $a_1 = b$.

The first is the case already considered and the triangle is isosceles always. In the second case there have to be distinguished further the cases in which (i) a_1 and b_1 meet AC and BC both produced beyond C or produced beyond A and B, and (ii) one bisector meets one of the sides produced beyond C and the other bisector meets the other side produced beyond A or B, as the case may be. In case (i) the triangle is isosceles, and in (ii) it is not (no proof given). In the third case the triangle is in general not isosceles.

In case (i) when a_1 and b_1 meet AC, BC produced beyond C, let

$$\alpha_1 > \beta_1.$$

Then $$p + \beta > q + \alpha.$$

Thus $$AE > BD \quad \text{and} \quad y > x.$$

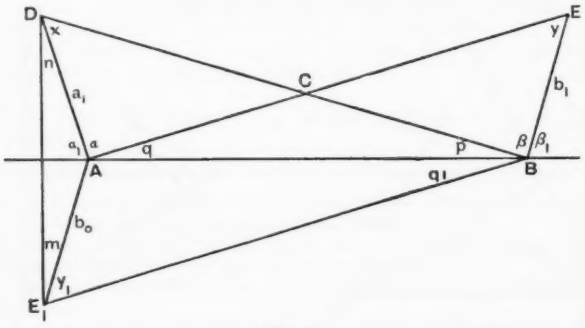


FIG. 3.

Now put AEB in the position BE_1A where $BE_1 = AE$, $q_1 = q$, $y_1 = y$, $b_0 = b_1 = a_1$, and join DE_1 (Fig. 3).

Then $$n = m \quad \text{and} \quad y_1 > x.$$

Thus $$m + y_1 > n + x.$$

Hence $BD > BE_1$ or $BD > AE$,

which contradicts the statement that $BD < AE$.

Thus $\alpha_1 = \beta_1$, and the triangle is isosceles.

Otherwise, if $\alpha_1 > \beta_1$, then $y > x$ as above, $p > q$, and so $AC > BC$. Then the triangles ACD, BCE can be inscribed in equal circles since $AD = BE$ and C is common; and, since $y > x$, we have $BC > AC$, which is contradictory. This proof is not so elementary, as it depends on the geometry of the circle.

When a_1 and b_1 meet CA, CB produced (Fig. 4) suppose that

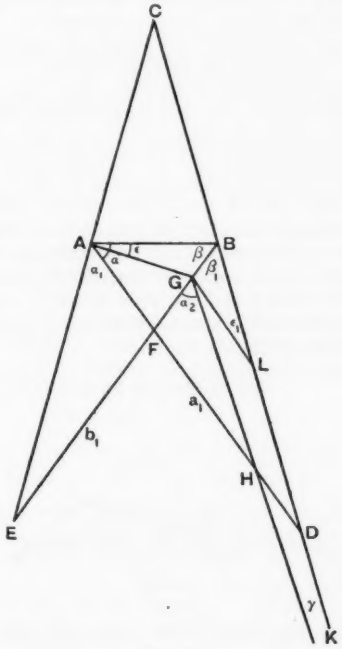


FIG. 4.

$\alpha > \beta$; then $BF > AF$ and $FD > FE$. Make $FG = FA$, $FH = FE$; then $GB = HD$.

The triangles HFG, EFA are congruent.

Thus $\alpha_2 = \alpha_1 = \alpha$,

and hence $\alpha_2 > \beta$.

Therefore GH meets CB beyond D at K at an angle γ easily seen to be equal to 2ϵ.

In the triangle DAC, $\alpha_1 = D + C$.

Thus $\alpha > D$, and so $BD > BA$.

Make $BL = BA$. Then the triangles BAG, BLG are congruent and $\epsilon_1 = \epsilon$.

But the exterior angle ϵ_1 of the triangle GLK is greater than γ. That is,

$$\epsilon > \gamma,$$

which contradicts the statement that $\gamma = 2\epsilon$. Thus $\alpha = \beta$, and the triangle is isosceles.

Steiner then says that the cases of Figs. 1 and 2 can be proved when $\alpha : \alpha_1 = \beta : \beta_1$; he next proves the theorem for the spherical triangle when the angles are bisected internally.

In the *Philosophical Magazine* for 1852 the problem is considered by J. J. Sylvester,* but with a different end in view. Sylvester was under the impression that the problem had appeared for the first time at Cambridge a year previously, and says that the solution is " by no means so obvious and self-evident as one would expect from the extreme simplicity of its enunciation ". He remarks also that " all the geometrical demonstrations yet given of this theorem are indirect ". The title of the paper is " On a simple geometrical problem illustrating a conjectured principle in the theory of geometrical method ". Sylvester's demonstration is as follows :

" Assuming that DAB is greater than EBA, it is easily seen that DE produced will cut BA at K on the side of it : also if AD and BE intersect in H, it is readily demonstrable, by a suitably constructed apparatus of similar triangles, that

$$AH : BH = CE : CD."$$

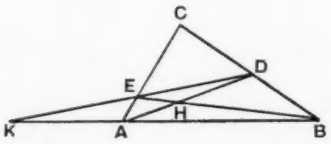


FIG. 5.

" But as HBA is less than HAB, AH is less than BH, and therefore CE is less than CD, and therefore CED is greater than CDE; that is to say, CAB less K is greater than CBA plus K, and therefore DAB less K is greater than CBA plus K; that is, ADE is greater than ABE, and therefore the perpendicular from A upon DE is greater than that from E on AB, which is easily proved to be absurd. Hence the triangle is proved to be isosceles. This proof, it is obvious, remains good for *all cases* in which EB and DA, drawn on either side of the base, divide the angles at the base proportionally, provided that these lines remain equal and make positive or negative angles with the base not *less* than one-half of the

* See also *Collected Works*, I (ed. H. F. Baker, Cambridge, 1904).

respective corresponding angles which the sides of the triangle are supposed to make with it. The analytical solution, as might be expected, extends the result still further. To obtain this, let

$$BAC = n \,.\, BAD, \quad ABC = n \,.\, ABE,$$

n for the present being any numerical quantity, positive or negative; calling $BAC = 2n\alpha$, $ABC = 2n\beta$, we readily obtain, by comparison of the equal dividing lines with the base of the triangle,

$$\frac{\sin(2n\alpha + 2\beta)}{\sin 2n\alpha} = \frac{\sin(2n\beta + 2\alpha)}{\sin 2n\beta},$$

or

$$\frac{\sin(2n\alpha + 2\beta)}{\sin(2n\beta + 2\alpha)} = \frac{\sin 2n\alpha}{\sin 2n\beta};$$

and by an obvious reduction,

$$\frac{\tan(n-1)(\alpha-\beta)}{\tan n(\alpha-\beta)} = \frac{\tan(n+1)(\alpha+\beta)}{\tan n(\alpha+\beta)}.$$

When this equation is put under an integer form, it is of course satisfied by making $\alpha = \beta$; on any other supposition than $\alpha = \beta$ it evidently cannot be satisfied by admissible values of the angles for any value of n between $+1$ and $+\infty$; for on that supposition, since $(\alpha - \beta)$ and $(\alpha + \beta)$ are each less than $180°/2n$, the first side of the equation will be necessarily a proper fraction and positive; but the second side, either a positive improper fraction if $(n+1)(\alpha+\beta)$ be less, and a negative proper or a negative improper fraction if $(n+1)(\alpha+\beta)$ be greater than a right angle.

If n be negative, let it equal $-m$, then

$$\frac{\tan(m+1)(\alpha-\beta)}{\tan m(\alpha-\beta)} = \frac{\tan(m-1)(\alpha+\beta)}{\tan m(\alpha+\beta)}$$

and for the same reason as before, if m lies between ∞ and 1, this equation cannot be satisfied. Hence the theorem is proved to be true for all values of n, except between $+1$ and -1. For these values it ceases to be true; in fact, for such values for any given value of $(\alpha - \beta)$ there will be always, as it may easily be proved, one or more values of $(\alpha + \beta)$; thus if $n = \frac{1}{2}$, the equation becomes

$$\frac{\tan \frac{3}{2}(\alpha+\beta)}{\tan \frac{1}{2}(\alpha+\beta)} = -1,$$

and if $n = -\frac{1}{2}$,

$$\frac{\tan \frac{3}{2}(\alpha-\beta)}{\tan \frac{1}{2}(\alpha-\beta)} = -1,$$

showing that $\alpha + \beta = 90°$ and $\alpha - \beta = \pm 90°$ in these respective cases will afford a solution over and above the solution $\alpha = \beta$, which is easily verified geometrically. . . . My reader will now be prepared to see why it is that all the geometrical demonstrations given of this theorem, even in the simplest case of all, namely when $n = 2$, are

indirect, I believe I may venture to say *necessarily* indirect. It is because the truth of the theorem depends on the necessary non-existence of real roots (between prescribed limits) of the analytical equation expressing the conditions of the question; and I believe that it may be safely taken as an axiom in geometrical method, that wherever this is the case, no other form of proof than that of *reductio ad absurdum* is possible in the nature of things. If this principle is erroneous, it must admit of an easy refutation in particular instances."

In the *Philosophical Magazine* for 1874 there is a letter from N. M. Ferrers, who attaches a letter received from a lady in Oakland, California, named Chart, who had seen Sylvester's paper and enclosed a *direct* solution found in 1842 by Mr. Hesse, a friend of hers. Ferrers had shown the solution to Sylvester; both of them agreed that it was sound and asked the *Phil. Mag.* to publish it: this was done. The solution is as follows:

In the triangle ABC the angles CAB, CBA are bisected and the portions of the bisecting lines AE, BD contained between the vertices A, B and the opposite sides BC, AC are equal: to prove that the angles α and β are also equal.

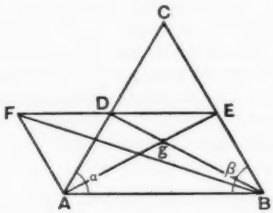


FIG. 6.

Find a point F so that $FA = AD$ and $FE = AB$ and draw FB.

I. We have from the equality of the triangles FAE, ADB,

$$AF = AD,$$
$$FE = AB,$$
$$AE = DB.$$

II. $$ADB + \tfrac{1}{2}\alpha = DgE,$$
$$BEA + \tfrac{1}{2}\beta = DgE.$$

Thus $$ADB + \tfrac{1}{2}\alpha = BEA + \tfrac{1}{2}\beta.$$

But $$ADB = FAE, \quad FEA = \tfrac{1}{2}\beta. \quad \text{.....(i)}$$

Hence $$FAE + \tfrac{1}{2}\alpha = BEA + FEA,$$

or $$FAB = BEF.$$

Also $$FB = FB, \text{ and } FE = AB.$$

III. From this follows the equality of the triangles *FBA*, *FBE*, considering the following equality of angles :

$$FAB = FEB = BEA + \tfrac{1}{2}\beta = DgE. \quad \text{.....(ii)}$$

But $DgE = 180° - \frac{1}{2}(\alpha + \beta)$;

and $\alpha + \beta < 180°$, and so $\frac{1}{2}(\alpha + \beta) < 90°$.

Therefore DgE or $FAB > 90°$.

From III it follows that $FA = BE$,

and by construction, $FA = AD$.

Therefore $BE = AD$,

from which results the equality of the triangles *ABE*, *ABD*, and hence $\alpha = \beta$.

The reader who is interested in the history of the problem will find a very interesting and exhaustive article by Dr. J. S. Mackay in *Proceedings of the Edinburgh Mathematical Society*, XX, 1901-1902.

J. P. McCarthy.

1201. . . . the reading in the Centigrade scale with 273 added is often called absolute temperature . . . but if we use the adjective absolute we are always liable to challenge from the thermodynamical expert and more than a single lecture would be required to explain how absolute temperature is lawfully obtained from a reading of the thermometer that one buys in a shop. So in self-defence we call the reading of the ordinary thermometer, amplified by the addition of 273 to the Centigrade scale, the tercentesimal temperature with *tt* as its symbol, and no degree mark, thereby placing ourselves on the side of the scientific angels who think dynamically, without pursuing all their intricate reasoning. For them at any rate a *negative* temperature is quite out of the question, a thing imagination boggles at.—Sir Napier Shaw, *The Drama of Weather*, pp. 100-101. [Per Mr. P. J. Harris.]

1202. *Undershaft* : . . . I offer you half [of the proposed terms] . . .

Cusins [disgusted] ; Half! . . .

Undershaft : . . . I will go a step further . . . I will give you three-fifths ; but that is my last word.

Cusins : Done! . . . By the way . . . I am a classical scholar, not an arithmetical one. Is three-fifths more than half, or less?—G. B. Shaw, *Major Barbara*, Act iii. [Per Mr. P. J. Harris.]

1203. The Aryan Angle.

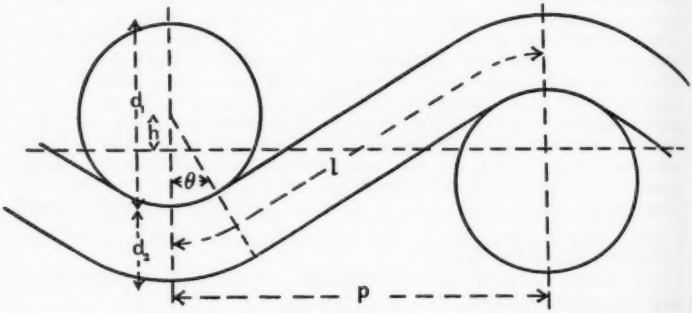
Babylonian scientists recognised that the three easiest angles to create were the right-angle (90°), the angle of an equilateral triangle (60°) and the half right-angle (45°), and sought a unit in which all three could be easily expressed. When navigation developed and calculation became a geometry of motion, it was seen that time entered into spatial relationship, and angle measurements were harmonised with the method of recording time. Superimposition of the three simple angles led to the adoption of a 90° value for the right-angle as most convenient, enabling most of the common angles to be stated as whole units, and giving a circle easily factorable by 24, the number of hour-divisions in the day.—*News Review*, Vol. iv, No. 18, p. 34. [Per Mr. G. L. Parsons.]

AN APPEAL.

It was announced in the May issue of the *Gazette*, that the General Teaching Committee of the Association has appointed a sub-committee to enquire further into the teaching of mathematics to technical students. Many teachers (especially those elementary and secondary teachers who also do part-time evening work) lack the time to explore the vast field of the application of mathematics to other subjects, the knowledge of which would, as they realise, do so much to increase the efficiency and interest of their work.

Nor is it only the teachers of technical students who are anxious to become aware of the ways in which mathematics is used, as a reading of the reports of discussions at the Annual Meeting of last January will show.

In his presidential address to the London Branch, Professor Hogben said, "A striking thing about contemporary education is the increasing demand for mathematical training. The educational problem which arises from the rapid mathematisation of science during the last half-century has found us unprepared and is largely an unsolved one. On that account the most signal contemporary contributions to educational technique are beginning to come from the ranks of mathematical teachers in technical institutes."



It therefore appears that an attempt to collect details of applications of elementary mathematics, over as wide a field as possible, would, if successful, prove of real assistance to almost all teachers of mathematics. The sub-committee has decided to issue this appeal, asking all who light upon such applications to be good enough to assist them in making this collection.

Two instances of the type of thing required may be given:

(1) Eddington's quadratic,

$$10x^2 - 136x + 1 = 0,$$

whose roots are proportional to the masses of the proton and the electron (*Relativity Theory of Protons and Electrons*, p. 219).

(2) The geometry and trigonometry of the accompanying figure, especially the relation between the "crimp" $(l/p - 1)$ and θ, taking d_1, d_2, and l as constants. (Peirce, "The Geometry of Cloth Structure," *Shirley Institute Memoirs*, xv, 1936, and Womersley, "The Application of Differential Geometry to the Study of the Deformation of Cloth under Stress," *ibid.* xvi, 1937). The figure represents the section of one weft and two warp threads. Incidentally, Womersley's use of the weft and warp threads as a basis for intrinsic co-ordinates is a similar type of application, though at a level higher than is reached in technical or secondary schools.

Will any reader of this appeal who comes across material which appears suitable for inclusion in the collection, please send an account of it (with, where possible, reference to the original source) to the secretary of the sub-committee,

A. BUXTON, 37 Heoldon, Whitchurch, Cardiff.

It is hoped that the response will justify the periodical publication of the material received, or at least such of it as is not already available in mathematical textbooks, in the *Gazette*.

W. G. BICKLEY
(Chairman of the sub-committee).

1204. By degrees mathematics asserted a paramountcy in the Schools which was inimical to every other branch of culture, and its own free development was thwarted by the extreme deference to Newton's laws, which were accepted as the final exposition of a divinely inspired creed. Under the influences of "mad Mathesis", as Pope had anticipated, the Muses lay "held in tenfold bonds". A Cambridge skit of 1774 says:

> See Euclid proudly spurns the Mantuan muse,
> While gentle Horace wipes Maclaurin's shoes.
> There Homer learns the theory of light,
> And tortured Ovid learns to sum and write.—

Gray, *Cambridge University* (1926), pp. 225-6.

1205. He was going to Sicily. . . . It was not merely a good poet like Theocritus nor a good engineer like Archimedes he was going out to search for. There were possibly poets and engineers as good within walking distance of Rickmansworth.—Louis Golding, *The Camberwell Beauty*, pp. 75-6.

1206. According to Jean Wier (1568) the number of demons is 7,409,127 "*apart from errors of calculation*". Later authorities identified this number as being of the order of six times the Pythagorean number 1234321, *i.e.* 7,405,926.—From a lecture. [Per Dr. J. Wishart.]

1207. It was in pre-war years that the idea of a free pendulum emerged from the chrysalis stage where it had lain hid since 1899, wrapped up by its parent as carefully as a cocoon is wrapped in silk.—F. Hope-Jones, *Electric Clocks*, p. 189, (London, 1931). [Per Mr. Frank Robbins.]

ON DIFFERENTIATION FROM FIRST PRINCIPLES.

By E. G. Phillips.

Questions on differentiating from first principles many of the elementary functions, such as $\sin x$, x^n, $\log x$, a^x, are still often set in examination papers on elementary Differential Calculus, in spite of the difficulties of evaluating rigorously the necessary limits.

At a meeting of the North Wales branch of the Mathematical Association last year, when the questions set in the Higher Certificate examination were under discussion, several schoolmasters mentioned the difficulty which they experience in teaching differentiation from first principles to beginners in the subject. I need hardly state that the method of treating this subject in the more old-fashioned textbooks gives grounds for much criticism. These treatments usually begin by the evaluation of several "important limits" such as $\lim_{x\to 0}(a^x-1)/x$, $\lim_{x\to\infty}(1+1/x)^x$, employing without hesitation infinite series, the "proof" running somewhat on these lines:

"We have $$a^x = 1 + x\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \dots;$$

hence $$\frac{a^x-1}{x} = \log_e a + x\ [\text{a convergent series}],$$

and the limit of the right-hand side as $x\to 0$ is clearly $\log_e a$."

In these days, when rigour in mathematical demonstration is the accepted thing, criticism of such methods scarcely needs making. One has only to remark "Suppose that the 'convergent series' has sum $1/x$" and the futility of such a "proof" becomes evident at once.

Authors of the newer textbooks on the Differential Calculus do not usually give proofs which come under criticism on the ground that they are unrigorous, but those teachers who experience difficulties may welcome a brief note on this subject.

If we use modern methods of teaching the Calculus, we realise that the only two elementary functions which have to be differentiated *from first principles* are x^n, when n is a positive integer, and $\sin x$.

For x^n, when n is a positive integer, the evaluation of the familiar limit only requires the binomial theorem for a positive integral index; since this does not require the use of infinite series, there is no difficulty. For the function x^α, when α is not a positive integer, the limit does of course involve the infinite series expansion of $(1+h/x)^\alpha$. Since however the derivative of x^α, when α is rational, can be readily deduced from that of x^n, there is clearly no reason for differentiating x^α *from first principles* at all.

For the derivative of $\sin x$, we cannot, of course, avoid the use of the limit of $\sin\theta/\theta$ as $\theta\to 0$. Since a completely rigorous discussion of the trigonometrical functions cannot be given to beginners at all, if we avoid *all* appeal to geometrical intuition as an element of proof,

the familiar geometrical proof that $\sin\theta<\theta<\tan\theta$ must be used to establish that $\lim_{\theta\to 0}(\sin\theta/\theta)=1$.

It may be remarked here that the rigorous definition of $\sin x$ as the sum-function of the infinite series $x-x^3/3!+x^5/5!-\ldots$, or the alternative method based on the equation

$$\arctan x=\int_0^x\frac{dt}{1+t^2},$$

as described by Hardy,* cannot possibly be used *with beginners* ; and so we must either use the familiar proof based on geometrical intuition or else dispense altogether with any attempt to prove that the derivative of $\sin x$ is $\cos x$.

The next question is that of the exponential and logarithmic functions. To differentiate from first principles the function e^x, we need, of course, to evaluate the limit of $(e^h-1)/h$ and the difficulty of using infinite series arises again. But if we use the modern method of introducing the logarithmic function, by defining $\log x$ as $\int_1^x dt/t$, a quite convincing treatment, involving only the concept of the definite integral as an area, enables us to discuss the function $\log x$ by means of the graph of $y=1/x$. For the details of the treatment on these lines reference may be made to a modern textbook on the Calculus.†

The result required, that the derivative of $\log x$ is $1/x$, readily follows, for, if x and $x+h$ are positive,

$$\log(x+h)-\log x=\int_1^{x+h}\frac{dt}{t}-\int_1^x\frac{dt}{t}=\int_x^{x+h}\frac{dt}{t},$$

from which we easily deduce that $\{\log(x+h)-\log x\}/h$ lies between $1/x$ and $1/(x+h)$, and so

$$\frac{d}{dx}(\log x)=\frac{1}{x}.$$

The exponential function can then be considered as the inverse function of the logarithm. Although the rigorous exposition requires the inverse function existence theorem, which is out of the question for beginners, a quite satisfactory geometrical discussion by means of the graphical representation of $\log x$ and e^x can be given : this leads readily to the result required, that

$$\frac{d}{dx}(e^x)=e^x.$$

Since $a^x=e^{x\log a}$, and the hyperbolic functions, $\sinh x$ and $\cosh x$, are linear combinations of e^x and e^{-x}, these functions present no further difficulty.

One of the definite advantages of introducing the logarithmic and

* *Pure Mathematics* (1925), p. 389.

† See, for example, Durell and Robson, *Elementary Calculus*, II, p. 251 *seq.*

exponential functions in this way is that, when the pupils reach the stage when they need a fully rigorous discussion of these functions, they will have nothing to unlearn.

If schoolmasters would teach the Differential Calculus on the lines briefly indicated here, it appears that the difficulties about the teaching of differentiation *from first principles* all disappear by the simple expedient of never teaching it at all except for x^n and $\sin x$. If, however, examiners still continue to ask for the differentiation *from first principles* of $\log x$, a^x, and even $\sin^{-1}x$, $\sqrt{(a^3+x^3)}$ (which I have actually seen), one cannot but sympathise with the schoolmasters. E. G. P.

1208. TRAFFIC MATHS.

When the New York police have to decide whether to install traffic lights at a crossing, they do it by means of the algebraical formula reproduced here.

I will not attempt to follow the calculation by which they arrive triumphantly at " composite intersection rating "—the figure which reveals how busy a crossing is. Thirteen different factors have to be determined before the sum can be worked out.

Intersection Rating

$$\left[\left(\frac{F_m \times F_c + \frac{(L_m + L_c)}{2}}{K}\right)\left(\frac{(S_h{}^2)}{S_c{}^2}\right)\right]$$

$$+$$

$$\left[\left(\frac{F_m \times P_m}{K}\right)\left(\frac{W_m}{W_k}\right)\right]$$

$$+$$

$$\left[\left(\frac{F_c \times P_c}{K}\right)\left(\frac{W_c}{W_k}\right)\right] + A = IR$$

or

Composite Intersection Rating

A Problem for New York Policemen

They include the vehicular flow on both streets at " peak hour ", the number of cars turning from one street into the other, the average speed of " hazardous " drivers, and the number of pedestrians who want to cross.

The vital factor " A ", added after everything else has been worked out, represents the policeman's commonsense. It is defined as " arbitrary value assigned for special conditions ".

London traffic authorities have the same problem. They solve it, however, without any algebra other than " A ".

Consultation between the police, the Ministry of Transport, and the local authority decides the location of traffic lights, although a rough vehicular census is sometimes taken for the information of the Ministry.—*Daily Telegraph.*

MATHEMATICAL NOTES.

1308. *The evaluation of definite integrals in terms of inverse functions.*

The note on the attraction of a spheroid (*Gazette*, xxii, p. 78) illustrates a practical detail that is often overlooked, namely, that even when an integral is of one of the standard forms, its discussion may be facilitated by the explicit use of the transformation by which the theoretical evaluation is effected. With u for $\sin\phi$, the final integral in Sir Frank Dyson's note is

$$\int_{-1}^{1} \frac{du}{R}$$

where

$$R^2 = z^2 + a^2 - 2bzu - a^2e^2u^2$$
$$= z^2/e^2 + a^2 - (bz/ae + aeu)^2.$$

Instead of quoting the integral function as an inverse sine, let us write down the appropriate substitution, namely,

$$bz/ae + aeu = q \sin\chi,$$

where $q^2 = z^2/e^2 + a^2$. We see at once that the value of the integral is $(\beta - \alpha)/ae$ where α, β are given by

$$aeq \sin\beta = bz + a^2e^2, \qquad q\cos\beta = z - b,$$
$$aeq \sin\alpha = bz - a^2e^2, \qquad q\cos\alpha = z + b,$$

the values of $q\cos\alpha$ and $q\cos\beta$ being simply the extreme values of R. With the values of the four functions $\sin\alpha$, $\cos\alpha$, $\sin\beta$, $\cos\beta$ in front of us, we can write down whatever trigonometrical function of $\beta - \alpha$ we wish. The simplest function is that which Sir Frank gives, namely, $\tan\frac{1}{2}(\beta - \alpha)$, which is $(\sin\beta - \sin\alpha)/(\cos\alpha + \cos\beta)$, and is equal to ae/z.

Since the integral, before its final evaluation, is expressed in terms only of $\sin\phi$, it is evident that the actual introduction of the eccentric angle can be avoided. We have, in fact, on the ellipse,

$$\frac{xdx}{-a^2} = \frac{ydy}{b^2} = \frac{xy(xdy - ydx)}{b^2x^2 + a^2y^2} = \frac{xyr^2d\theta}{a^2b^2},$$

whence $r^2xd\theta = a^2dy$, giving directly

$$\int_{-\pi/2}^{\pi/2} \frac{r^2xd\theta}{R} = a^2\int_{-b}^{b} \frac{dy}{R},$$

with $R^2 = (z-y)^2 + a^2(1 - y^2/b^2)$.

The principle of retaining the substitution is especially useful with inverse hyperbolic functions, if a reduction to logarithms or a numerical evaluation is required. For example, if we quote a formula and write down

$$\int_1^2 \frac{dx}{\sqrt{(3x^2 + 4x + 5)}} = \frac{1}{\sqrt{3}}\left(\operatorname{arg\,sh}\frac{7}{\sqrt{11}} - \operatorname{arg\,sh}\frac{5}{\sqrt{11}}\right),$$

most of the chances of error are still ahead of us, but if we put down our substitution, $3x+2=\sqrt{11}\,\mathrm{sh}\,\theta$, $\theta\geqslant 0$, we have the value of the integral as $(\beta-\alpha)/\sqrt{3}$, where not only are the hyperbolic sines given by the initial and final values of $(3x+2)/\sqrt{11}$, but it is implicit in the use of the substitution that the corresponding hyperbolic cosines are given by the initial and final values of

$$\sqrt{(9x^2+12x+15)}/\sqrt{11}\,;$$

that is,

$$\sqrt{11}\,\mathrm{sh}\,\beta=8,\quad \sqrt{11}\,\mathrm{ch}\,\beta=5\sqrt{3},\quad \sqrt{11}\,\mathrm{sh}\,\alpha=5,\quad \sqrt{11}\,\mathrm{ch}\,\alpha=6,$$

implying

$$e^{\beta}=\frac{8+5\sqrt{3}}{\sqrt{11}},\quad e^{\alpha}=\sqrt{11},$$

and therefore

$$\beta-\alpha=\log_e\frac{8+5\sqrt{3}}{11}.$$

Similarly if the square roots of two linear functions are involved, we rationalise these explicitly. For example, to evaluate the integral

$$I=\int_2^3\sqrt{\frac{2x-3}{3x-1}}\,dx$$

we adjust the coefficients of x to be equal, and write

$$6x-9=7\,\mathrm{sh}^2\,\theta,\quad 6x-2=7\,\mathrm{ch}^2\,\theta,\quad \theta\geqslant 0.$$

Then

$$I=\frac{7\sqrt{2}}{3\sqrt{3}}\int_{\alpha}^{\beta}\mathrm{sh}^2\,\theta\,d\theta$$

where

$$\sqrt{7}\,\mathrm{sh}\,\beta=3,\quad \sqrt{7}\,\mathrm{ch}\,\beta=4,\quad \sqrt{7}\,\mathrm{sh}\,\alpha=\sqrt{3},\quad \sqrt{7}\,\mathrm{c}\quad \alpha=\sqrt{10},$$

and since

$$\int\mathrm{sh}^2\,\theta\,d\theta=\tfrac{1}{4}\int(e^{\theta}-e^{-\theta})^2\,d\theta=\tfrac{1}{8}(e^{2\theta}-e^{-2\theta})-\tfrac{1}{2}\theta=\tfrac{1}{2}(\mathrm{sh}\,\theta\,\mathrm{ch}\,\theta-\theta),$$

we have

$$I=\frac{1}{3\sqrt{6}}\left\{12-\sqrt{30}-7\log\frac{7}{\sqrt{3}+\sqrt{10}}\right\}$$

with nothing to memorise and little risk of slips in the working.

E. H. N.

1309. *The contact of a variable circle with two fixed circles.*

Let α, β be two fixed oriented circles in a plane, with centres A, B and radii a, b; let ρ be a variable oriented circle, with centre P and radius r, which touches both α and β, and let the points of contact be U, V. The circles being oriented, the radii a, b, r have definite signs, and since contact of oriented circles means not only

contact of the circumferences, but identity of direction at the point of contact, the distances UA, UP, in the direction at right angles to the common tangent at U, are a, r, and the distances VB, VP, in the direction at right angles to the common tangent at V, are b, r. It follows that the powers Π_α, Π_β of P for α, β are given unambiguously by

$$\Pi_\alpha = r(r-2a), \quad \Pi_\beta = r(r-2b),$$

and therefore that the difference between these powers is given by

$$\Pi_\beta - \Pi_\alpha = 2(a-b)r.$$

But as P varies, the difference $\Pi_\beta - \Pi_\alpha$ is proportional to the distance of P from the radical axis of α and β; to be precise, let a direction be chosen for measurements parallel to the line of centres AB, and let c be the length of AB in this direction; then if the line through P parallel to AB cuts the radical axis γ of α and β in K,

$$\Pi_\beta - \Pi_\alpha = 2c \,.\, PK.$$

Hence $r = e \,.\, PK$, where $e = c/(a-b)$:

If a variable directed circle ρ touches two fixed directed circles α, β, the radius of ρ is proportional to the distance of its centre from the radical axis of α, β.

Further, the lengths AP, BP are $r-a$, $r-b$; let λ, μ be lines parallel to γ, such that the distances from them to γ have the fixed values a/e, b/e, and let PK cut these lines in L, M. Then

$$PL = PK - a/e, \quad PM = PK - b/e,$$

and therefore

$$r - a = e \,.\, PL, \quad r - b = e \,.\, PM:$$

The locus of the centre of a variable directed circle which touches two fixed directed circles satisfies the focus-directrix definition of a conic simultaneously for the centres of the two fixed circles, the eccentricity with respect to each focus being the ratio of the distance between their centres to the difference between their radii.

Whether the conic is an ellipse or a hyperbola depends on whether the absolute value of the algebraical difference between the radii of the fixed circles is greater than or less than the absolute value of the distance between their centres; the absolute value of the algebraical difference is the difference or the sum of the arithmetical radii according as the orientation of the circles is like or unlike.

Given two undirected circles, direction can be assigned to them in four ways, but simultaneous reversal of their direction does not affect the locus of the centre of a circle touching them both: if the circle of radius r with centre P touches circles of radii a, b round A, B, then the circle of radius $-r$ with the same centre P touches circles of radii $-a$, $-b$ round A, B. Thus with undirected circles,

the locus of the centre of a variable circle which touches two fixed circles is a pair of conics, one with eccentricity $|c|/(|a|\sim|b|)$, and one with eccentricity $|c|/(|a|+|b|)$. If one circle is inside the other, the distance $|AB|$ is smaller than the difference between the arithmetical radii and both conics are ellipses. If each circle is outside the other, $|AB|$ is greater than the sum of the arithmetical radii and both conics are hyperbolas. If the circles intersect, the distance between their centres is between the difference and the sum of their radii: one conic is an ellipse and the other a hyperbola. Two special cases may be mentioned: if the circles are equal, one conic degenerates into the radical axis; if one circle degenerates to a point, the two conics coincide.

The results in this note are classical, but I do not remember to have comes across a line of approach, to this or any other problem, in which the focus-directrix property became apparent at the two foci simultaneously. E. H. N.

1310. *Note on a certain identity.*

The identity

$$(\phi+\psi)^n=(\phi^n+\psi^n)+n(\phi^{n-1}+\psi^{n-1})+\frac{n(n+1)}{1\,.\,2}(\phi^{n-2}+\psi^{n-2})$$
$$+\ldots+\frac{n(n+1)\ldots(2n-2)}{1\,.\,2\ldots(n-1)}(\phi+\psi), \quad \ldots\ldots\ldots\ldots(1)$$

where ϕ and ψ satisfy the relation $\phi+\psi=\phi\psi$, and n is any positive integer, is particularly useful for deriving trigonometric formulae, and in proving interpolation formulae with symmetrical differences.

The following proof illustrates a use of contour integration.

Denoting $\dfrac{n(n+1)\ldots(n+s-1)}{1\,.\,2\ldots s}$ by $\begin{bmatrix} n \\ s \end{bmatrix}$, let

$$s(\phi)\equiv\phi^n+\begin{bmatrix} n \\ 1 \end{bmatrix}\phi^{n-1}+\ldots+\begin{bmatrix} n \\ n-1 \end{bmatrix}\phi. \quad \ldots\ldots\ldots\ldots(2)$$

The coefficients $\begin{bmatrix} n \\ s \end{bmatrix}$ are the coefficients of x^s in $(1-x)^{-n}$; hence

$s(\phi)=$ coefficient of x^{n-1} in

$$\phi\left\{1+\begin{bmatrix} n \\ 1 \end{bmatrix}x+\begin{bmatrix} n \\ 2 \end{bmatrix}x^2+\ldots\begin{bmatrix} n \\ n-1 \end{bmatrix}x^{n-1}+\ldots\right\}\{1+\phi x+\phi^2x^2+\ldots\}$$

$$=\text{coefficient of } x^{n-1} \text{ in } \frac{1}{(1-x)^n}\cdot\frac{\phi}{(1-\phi x)}$$

$$=\text{coefficient of } x^{n-1} \text{ in } \frac{1}{(1-x)^n}\cdot\frac{1}{(p-x)},$$

where $p=1/\phi$.

The above series are absolutely convergent, for all values of n, provided $|x|<1$, and also $|x|<|p|$.

Similarly $s(\psi)$ = coefficient of x^{n-1} in $\dfrac{1}{(1-x)^n}\dfrac{1}{(q-x)}$, for $|x|<|q|$, where $q=1/\psi$.

Hence

$$s(\phi)+s(\psi)=\text{residue at } x=0 \text{ of } \frac{1}{x^n(1-x)^n}\left(\frac{1}{p-x}+\frac{1}{q-x}\right). \quad \ldots\ldots(3)$$

If η is the smallest of 1, $|p|$, $|q|$, then as $x\to 0$ we may take $|x|<\eta$, and hence the above multiplication of series is permissible.

Consider the integral

$$I=\frac{1}{2\pi i}\int_c \frac{1}{x^n(1-x)^n}\left(\frac{1}{p-x}+\frac{1}{q-x}\right)dx,$$

where the contour C is such that it encloses the points $x=0$ and $x=p$ and does not contain either of the points $x=1$, $x=q$.

If we make the transformation $x=1-y$, then

$$p-x=-(q-y)\,; \quad q-x=-(p-y), \text{ since } p+q=1.$$

Hence
$$I=\frac{1}{2\pi i}\int_{c'} \frac{1}{y^n(1-y)^n}\left(\frac{1}{p-y}+\frac{1}{q-y}\right)dy, \quad \ldots\ldots\ldots\ldots(4)$$

where the contour C' encloses the points $x=1$ and $x=q$ and does not contain either of the points $x=0$, $x=p$.

Now let the residues at $x=0$, p, q, 1 be $R(0)$, $R(p)$, $R(q)$, $R(1)$ respectively, then by (3) and (4)

$$R(0)+R(p)=R(1)+R(q). \quad \ldots\ldots\ldots\ldots(5)$$

If C'' is a large contour which encloses C and C', then provided $2n+1>1$, i.e. provided $n>0$, the integral round an infinite circle vanishes and thus $I_{c''}=0$.

Hence
$$R(0)+R(p)+R(q)+R(1)=0. \quad \ldots\ldots\ldots\ldots(6)$$

Combining (5) and (6),

$$R(0)=-R(p)=1/p^nq^n,$$

whence
$$\begin{aligned} s(\phi)+s(\psi)=R(0)&=1/p^nq^n \\ &=(\phi\psi)^n \\ &=(\phi+\psi)^n. \end{aligned}$$

We may remove the restriction (2) by writing

$$\phi=k\alpha,\ \psi=k\beta, \text{ then } \alpha+\beta=k\alpha\beta,$$

then
$$\begin{aligned} (\alpha+\beta)^n \equiv (\alpha^n+\beta^n)&+\begin{bmatrix} n \\ 1 \end{bmatrix}(\alpha^{n-1}+\beta^{n-1})\left(\frac{\alpha\beta}{\alpha+\beta}\right) \\ &+\begin{bmatrix} n \\ 2 \end{bmatrix}(\alpha^{n-2}+\beta^{n-2})\left(\frac{\alpha\beta}{\alpha+\beta}\right)^2 \\ &+\ldots+\begin{bmatrix} n \\ n-1 \end{bmatrix}(\alpha+\beta)\left(\frac{\alpha\beta}{\alpha+\beta}\right)^{n-1}. \end{aligned}$$

where α and β are now independent. W. R. ANDRESS.

1311. *A proof of the identity in Note* 1310.

The following is a direct proof, by elementary algebra, of the identity given by Mr. Andress in the preceding note.

Writing $$\frac{n(n+1)(n+2)\ldots(n+s-1)}{1.2.3\ldots s} \equiv \begin{bmatrix} n \\ s \end{bmatrix},$$

it is immediately verified that

$$\begin{bmatrix} n \\ r+1 \end{bmatrix} - \begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r+1 \end{bmatrix}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

We may make the convention that $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$, since this agrees with the difference relation (1).

It then follows that

$$\sum_{s=1}^{r} \begin{bmatrix} n \\ s \end{bmatrix} = \sum_{s=1}^{r} \left\{ \begin{bmatrix} n+1 \\ s \end{bmatrix} - \begin{bmatrix} n+1 \\ s-1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} n+1 \\ r \end{bmatrix} - 1,$$

and so $$\sum_{s=0}^{r} \begin{bmatrix} n \\ s \end{bmatrix} = \begin{bmatrix} n+1 \\ r \end{bmatrix}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

If now r is any positive integer, and if $x+y=xy$, then

$$(x+y)(x^r+y^r) = (x^{r+1}+y^{r+1}) + xy(x^{r-1}+y^{r-1})$$

$$= (x^{r+1}+y^{r+1}) + (x+y)(x^{r-1}+y^{r-1}).$$

By repeated application, we have

$$(x+y)(x^r+y^r) = (x^{r+1}+y^{r+1}) + (x^r+y^r) + (x+y)(x^{r-2}+y^{r-2})$$

$$= \ldots .$$

$$= (x^{r+1}+y^{r+1}) + (x^r+y^r) + (x^{r-1}+y^{r-1}) + \ldots + (x+y)(x+y)$$

$$= (x^{r+1}+y^{r+1}) + (x^r+y^r) + (x^{r-1}+y^{r-1}) + \ldots + (x^2+y^2) + 2(x+y).$$

Writing $x^r+y^r \equiv z_r$, we have the identity

$$z_1 z_r = z_{r+1} + z_r + z_{r-1} + z_{r-2} + \ldots + z_2 + 2z_1. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

We can now prove the main identity, viz. that

$$z_1^n = \sum_{r=0}^{n-1} \begin{bmatrix} n \\ r \end{bmatrix} z_{n-r}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

by induction.

Assuming it true for n, then

$$z_1^{n+1} = \sum_{r=0}^{n-1} \begin{bmatrix} n \\ r \end{bmatrix} z_1 z_{n-r},$$

whence, from (3),

$$z_1^{n+1} = z_{n+1} + z_n + z_{n-1} + z_{n-2} + \ldots + z_2 + 2z_1$$
$$+ \begin{bmatrix} n \\ 1 \end{bmatrix} \{z_n + z_{n-1} + z_{n-2} + \ldots + z_2 + 2z_1\}$$
$$+ \begin{bmatrix} n \\ 2 \end{bmatrix} \{z_{n-1} + z_{n-2} + \ldots + z_2 + 2z_1\}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \begin{bmatrix} n \\ n-1 \end{bmatrix} \{z_2 + 2z_1\}.$$

Summing by columns, and using (2), we have that

$$z_1^{n+1} = z_{n+1} + \begin{bmatrix} n+1 \\ 1 \end{bmatrix} z_n + \begin{bmatrix} n+1 \\ 2 \end{bmatrix} z_{n-1}$$
$$+ \ldots + \begin{bmatrix} n+1 \\ n-1 \end{bmatrix} z_2 + 2 \begin{bmatrix} n+1 \\ n-1 \end{bmatrix} z_1.$$

But it is easily verified that $2 \begin{bmatrix} n+1 \\ n-1 \end{bmatrix} = \begin{bmatrix} n+1 \\ n \end{bmatrix}$, and thus

$$z_1^{n+1} = \sum_{r=0}^{n} \begin{bmatrix} n+1 \\ r \end{bmatrix} z_{n+1-r},$$

which is (4) with $n+1$ in place of n. Since (4) is true for $n=1$, it follows that it is true for all positive integral values of n.

H. A. HAYDEN.

1312. *Comment on Note* 1229.

In Note 1229, A. A. Krishnaswami Ayyangar gave two apparently disconnected solutions of the problem :

To find a triangle PQR inscribed in another triangle ABC, with P, Q, R respectively on BC, CA, AB, such that the orthocentre of the inscribed triangle is the circumcentre of the outer one.

The present note tackles the problem analytically, and obtains the solutions from a quadratic equation.

Take the triangle ABC as triangle of reference in areal coordinates, and suppose that the coordinates of P, Q, R are respectively proportional to $(0, 1, \lambda)$, $(\mu, 0, 1)$, $(1, \nu, 0)$. It is clear that the coordinates of the circumcentre O are proportional to $(a \cos A, b \cos B, c \cos C)$.

The line QR is

$$-\nu x + y + \mu\nu z = 0,$$

and the line OP is

$$x(c \cos C - \lambda b \cos B) + y\lambda a \cos A - za \cos A = 0.$$

Now the lines (l_1, m_1, n_1), (l_2, m_2, n_2) are perpendicular if

$$a^2l_1l_2 + b^2m_1m_2 + c^2n_1n_2 - bc(m_1n_2 + m_2n_1)\cos A - ca(n_1l_2 + n_2l_1)\cos B - ab(l_1m_2 + l_2m_1)\cos C = 0.$$

Substituting, and using the relations such as

$$\cos A + \cos B \cos C = \sin B \sin C,$$
$$\cos^2 B - \cos^2 C = \sin^2 C - \sin^2 B,$$

we obtain as the condition that OP shall be perpendicular to QR

$$\lambda\mu\nu(a^2 - b^2) - c^2\mu\nu + a^2\nu\lambda + b^2\lambda - a^2\nu + (c^2 - a^2) = 0.$$

Similarly the conditions that OQ, OR shall be respectively perpendicular to RP, PQ are

$$\lambda\mu\nu(b^2 - c^2) - a^2\nu\lambda + b^2\lambda\mu + c^2\mu - b^2\lambda + (a^2 - b^2) = 0,$$
$$\lambda\mu\nu(c^2 - a^2) - b^2\lambda\mu + c^2\mu\nu + a^2\nu - c^2\mu + (b^2 - c^2) = 0,$$

where, by addition, it is clear that two of the conditions are linearly independent.

Now eliminate λ between the first two equations. We obtain an equation in μ, ν which is found to factorise, giving

$$\{\nu(\mu c^2 + a^2) + b^2\}\{\nu(\mu b^2 - \mu c^2 - a^2) + (\mu a^2 - b^2 + c^2)\} = 0.$$

We thus obtain the two sets of solutions (which can be shown to be grouped as follows) :

(i) $$\mu\nu c^2 + \nu a^2 + b^2 = 0,$$
$$\nu\lambda a^2 + \lambda b^2 + c^2 = 0,$$
$$\lambda\mu b^2 + \mu c^2 + a^2 = 0\ ;$$

(ii) $$\mu\nu(b^2 - c^2) + (\mu - \nu)a^2 - (b^2 - c^2) = 0,$$
$$\nu\lambda(c^2 - a^2) + (\nu - \lambda)b^2 - (c^2 - a^2) = 0,$$
$$\lambda\mu(a^2 - b^2) + (\lambda - \mu)c^2 - (a^2 - b^2) = 0.$$

Consider first solution (i). The third equation of the set can be obtained from the first two on eliminating ν. On the other hand, by eliminating a^2, b^2, c^2 between the three equations, we find the condition $\lambda\mu\nu = 1$, which shows that the lines AP, BQ, CR are concurrent, in a point whose coordinates satisfy the relations $z = \lambda y$, $x = \mu z$, $y = \nu x$. We can take the coordinates of this point to be $(1, \nu, \nu\lambda)$.

Now the first of the relations can be written in the form

$$a^2\nu \,.\, \nu\lambda + b^2\nu\lambda \,.\, 1 + c^2 \,.\, 1 \,.\, \nu = 0,$$

so that the point of concurrence lies on the circumcircle

$$a^2yz + b^2zx + c^2xy = 0.$$

It is clear that the coordinates of this point may be taken proportional to

$$(-a^2\lambda, b^2\lambda + c^2, b^2\lambda^2 + c^2\lambda),$$

so that there is an infinity of solutions depending on the parameter λ.

Now consider solution (ii). By means of Ptolemy's theorem it is easy to show that the first equation is precisely the condition that

the circle AQR should pass through the point O, and similarly for the other two. It is clear, as can be verified from the geometry, that any one of these equations is a consequence of the other two. Further, we again obtain an infinity of solutions which can be made to depend on the parameter λ.

It is hoped that these results may help towards a statement of the solution in terms of pure geometry. E. A. MAXWELL.

1313. *Postscript to Note* 1229 (*Mathematical Gazette*, Feb. 1937).

1. The problems suggested in the last lines of § 3 of my previous Note, 1229, are but generalisations of those given earlier. So a solution of the latter gives, *ipso facto*, a solution of the former, and furnishes besides a uniform method which comprehends all the alternative cases, instead of dealing with each of them by a special device. The alternative cases exhibit the phenomenon of *complete individuality* on account of the fact that they are derived from the *rational* roots of the determining equations.* In general, it may be said that any algebraic equation which solves a geometrical problem must, under certain circumstances, have all or some roots rational, and in such cases, since there is no such organic connection among two or more rational roots, as, for example, there is between two conjugate complex roots or quadratic surds, there is no compelling necessity for uniting them in a single geometric view. Thus in the theorem (α)†, either of the two cases does not suggest the existence of the other. Each case seems to live in a world of its own, completely suppressing the common ancestry.

2. Regarding the problems proper, we notice that reciprocation with respect to a circle simplifies them considerably for analytical methods. Using the given point which is to play the rôle of orthocentre, incentre, or excentre, as the centre of a base-circle for reciprocation, we reduce the problems to the following :

Given the base BC in position and two points X, Y on the sides AB, AC respectively of a triangle, construct it so as to have a given point T as (I) *orthocentre*, (II) *circumcentre*.

Take BC as the x-axis and the perpendicular from T to BC as the y-axis, and let T, X, Y be respectively $(0, a)$, (x_1, y_1), and (x_2, y_2).

The vertex A in problem I lies on the y-axis and may be taken as $(0, \rho)$, while the vertices B, C in problem II are equidistant from the origin and may, therefore, be represented by $(-\sigma, 0)$, $(\sigma, 0)$.

The equations for ρ and σ are easily seen to be

$$\rho^2 - \rho(y_1 + y_2 - x_1x_2/a) + y_1y_2 = 0 \quad \ldots\ldots\ldots\ldots (1)$$

and

$$\sigma^3 + \sigma^2(x_1 - x_2) + \sigma(a \cdot \overline{y_1 + y_2} - x_1x_2 - y_1y_2) + a(x_1y_2 - x_2y_1) = 0. \quad \ldots\ldots\ldots\ldots (2)$$

* Vide *infra*.

† Note 1229, *Math. Gazette*, vol. xxi, No. 242, p. 58.

Now no one can suspect that these equations will give rational roots when the base-line BC contains the orthocentre of the triangle TXY in problem II, and in the case of problem I, the third vertex of the triangle whose other two vertices are X, Y and whose incentre or excentre is T according as $X\hat{T}Y$ is obtuse or acute ! But theorems (α) and (β) * make this clear, though the analytical verification by means of the above equations is tedious. Thus there is a very small chance of inferring the existence of rational solutions given by the cases (α), (β) whose reciprocals are noted above. Is it not, therefore, a pure geometrical accident that there are such " queer alternatives "?

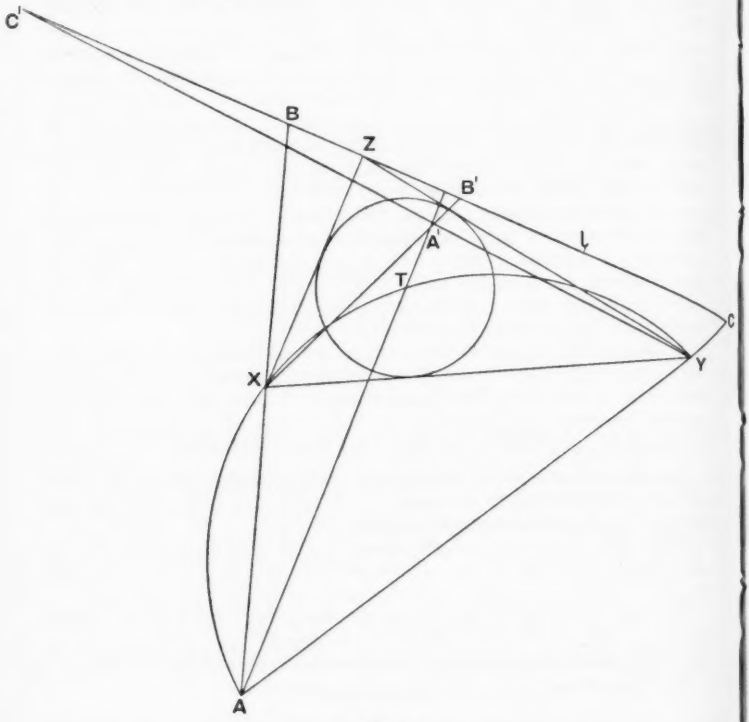


FIG. 1.

It must be possible to devise some elegant geometrical constructions to solve (1) by ruler and compasses and (2) by means of conic sections and adapt them to our problems, so as to get all the alter-

* *loc. cit.*, p. 58.

native cases *at one stroke*. These constructions evidently suffice also for the particular cases where the equations admit of rational roots.

The special constructions, however, for these particular cases are noteworthy for their extreme simplicity. The special problems may be restated as follows :

(A) XYZ is a triangle with T as an incentre (or excentre) ; l is a given straight line through Z. Required to draw two straight lines through X and Y so as to form a triangle with the given line l, such that T may be the orthocentre of this triangle.

There are *two* solutions :

(i) Draw the circle TXY to cut the perpendicular from T to l again in A. Join AX, AY to cut l in B, C respectively. Then the orthocentre of ABC is T.

(ii) Find the pole A' of l with respect to the incircle (or excircle) of XYZ. Join $A'X$, $A'Y$ to cut l in B', C'. Then T is the orthocentre of $A'B'C'$.

(B) XYZ is a triangle whose orthocentre is T * ; l is any given line through Z. Required to draw two straight lines through X and Y so as to form a triangle with l such that its circumcentre is T.

There are *three* solutions, of which two may become imaginary.

(i) Draw the circles TXZ, TYZ to cut l again in B_1 and C_1. Join B_1X, C_1Y to meet in A_1. Then T is the circumcentre of $A_1B_1C_1$.

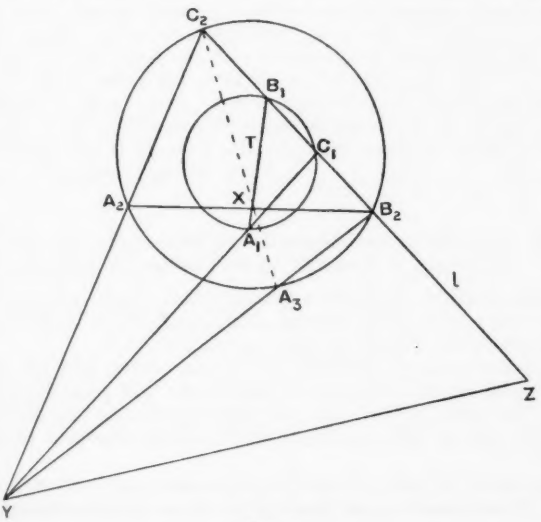


FIG. 2.

* This is the same as saying that TXY is a triangle whose orthocentre is Z. (*Vide* p. 59 above.)

(ii) and (iii). Draw the polar circle of XYZ to cut l in B_2 and C_2. Join B_2X, C_2Y to meet in A_2, and join C_2X, B_2Y to meet in A_3. There are two triangles formed, both of which have the same circumcentre T. In fact, the polar circle itself is the circumcircle of these triangles.

The solutions (ii) and (iii) will become imaginary when either the triangle XYZ is not obtuse-angled or the given l fails to cut the real polar circle.

3. An additional point of interest may be noted. It is well known that, in general, two triangles can be inscribed in a circle, such that its three sides pass respectively through three given points.* But if the three given points form a self-conjugate triangle with respect to the circle, an infinite number of triangles can be so inscribed. Conversely, if more than two triangles can be inscribed in a given circle and circumscribed about a given triangle, the latter triangle is self-polar to the circle. Again, if the given circle is concentric with the polar circle of the given triangle, the two triangles which can be inscribed in the given circle and circumscribed about the given triangle are congruent to each other and similar to the given triangle. For of the two possibilities envisaged by our theorems † (α) (1) and (2), the latter is ruled out and the former requires that the required triangle should be similar to the given triangle.

Reciprocally, in general, only two triangles can be inscribed in a given triangle so as to circumscribe a given circle and these become congruent being similar to the pedal triangle of the given triangle, when the given circle is concentric with the polar circle of the given triangle. But when the given circle is the polar circle itself, an infinite number of such triangles exist. Conversely, if more than two triangles can be drawn so as to circumscribe a given circle and be inscribed in a given triangle, the triangle is self-polar to the circle.

A. A. KRISHNASWAMI AYYANGAR.

1314. *A suggestion for bridging the gulf between the queer alternatives mentioned in Note* 1229 *of the* Gazette *for February* 1937.

It is required to construct a triangle PQR, whose vertices shall lie on the sides of a given triangle ABC, and which shall have the circumcentre, O, of ABC as orthocentre.

The polar circle of PQR will have O as centre, and the sides of PQR, which is self-conjugate to this circle, will pass through α, β and γ, the poles of BC, CA, AB with respect to this circle. For P is the pole of QR, and lies on BC. Hence the pole of BC lies on QR.

Also, since βP and γP are conjugate lines with respect to this circle, P lies on the conic which is the locus of the intersections of

* *Vide* Casey's *Sequel to Euclid*, p. 138 (4) [1886 edition]. Unfortunately, I cannot think of any more recent geometry textbook which deals with this problem.

† *loc. cit.*, p. 58 above.

conjugate lines through β and γ. Hence P is an intersection of this conic with BC. We shall call this conic Σ.

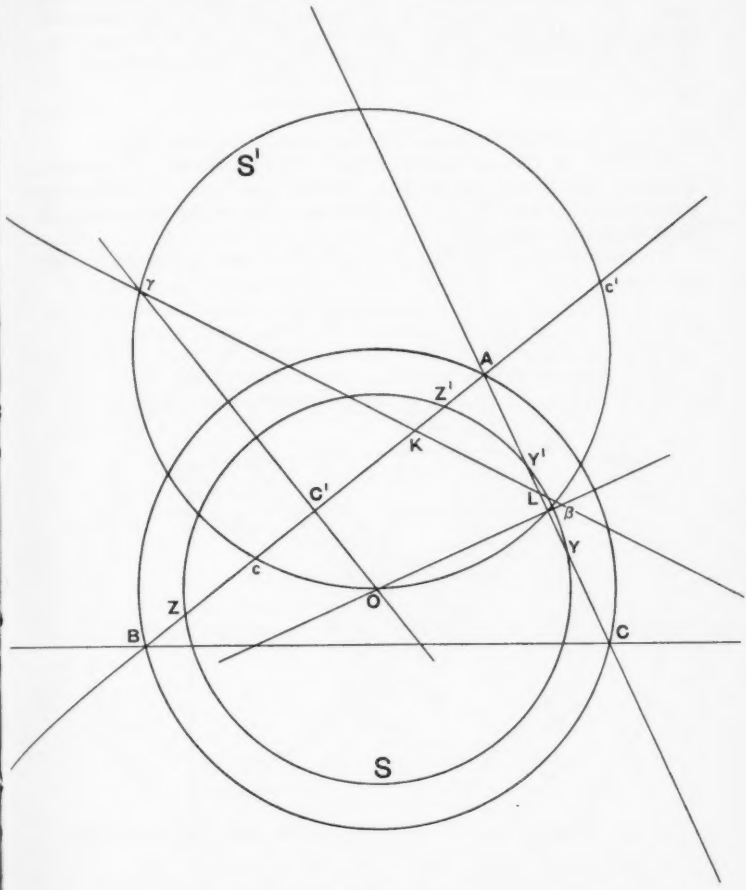


FIG. 1.

Let S be the polar circle of the proposed triangle, and let S cut AB at Z, Z' and AC at Y, Y'. Then Σ passes through β, γ, Y, Y', Z, Z'. Also let S' be the circle $O\beta\gamma$.

We shall show that this conic passes through the intersections of BC with the circle S'. For all conics through the intersections of S' and Σ cut AB in an involution which has Z, Z' and the inter-

sections of the circle $O\beta\gamma$ with AB (say c and c') as point-pairs. Further, since $C'Z \cdot C'Z' = C'O \cdot C'\gamma$ ($OZZ'\gamma$ are concyclic) and $C'O \cdot C'\gamma = C'c \cdot C'c'$, it follows that C' is the centre of the involution. Now if K is the point of intersection of $\beta\gamma$ and AB, then K, B form a point-pair of this involution, for γ, K, O, B are concyclic, whence $C'K \cdot C'B = C'O \cdot C'\gamma$. ($\gamma, K, O, B$ are concyclic since

$$\angle BO\gamma = \angle C = \angle \gamma KB,$$

$\beta\gamma$ being parallel to the side EF of the pedal triangle of ABC.)

Similarly, in the involution formed by the system $S'\Sigma$ on AC, L and C will form a point-pair. Hence the conic of the system through K will pass through B, and since K is on $\beta\gamma$, this conic will pass through L and C.

We have thus proved that BC and $\gamma KL\beta$ form a line-pair of the

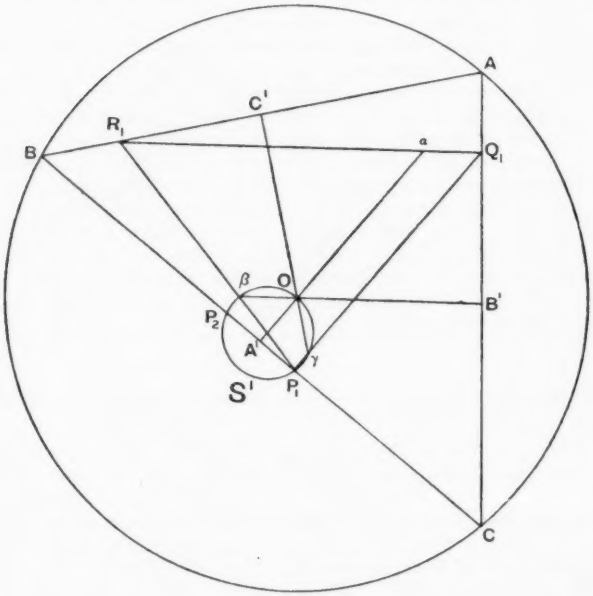


Fig. 2.

four-point system $S'\Sigma$, whence it follows that the positions of P are the intersections of BC with the circle S'. Thus $\angle \beta P\gamma = \angle A$, each being supplementary to $\angle \beta O\gamma$. Thus PQR is, in general, similar to ABC, when S is given.

Now Σ is, in general, a conic, but in the case where S is the

circumcircle of ABC, then Y', Z' coincide at A, Y with C and Z with B. Thus Σ degenerates into the line-pair $\beta A\gamma$, BC, and *every* point in BC is a possible position for P, and PQR is self-polar to the circumcircle of ABC.

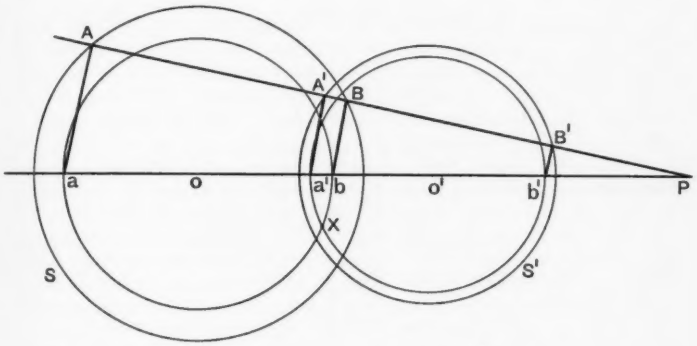
NOTE. It is not possible to draw a figure in which both S and the intersections of BC with the circle $O\beta\gamma$ are all real. In Fig. 1, S is real and the intersections imaginary; in Fig. 2, S is imaginary and the intersections are real. W. J. HODGETTS.

1315. *On Note* 1287.

The construction of a line which shall be trisected by its intersections with two circles, given in Note 1287 of the *Gazette*, is a special case of a more general construction depending on the following result, due to Professor P. K. Kashikar.

If a line cut two circles S and S' at A, B, A' and B' and the perpendiculars to it at these points cut the line of centres at a, b, a' and b', then P, the point of intersection of the line with the line of centres, lies on the circle coaxal with S, S', and also with the circles on ab, $a'b'$ as diameters. Let these circles intersect at X.

(In proving this result the power of K with respect to S will be denoted by k, etc.)



Let O and O' be the centres of S and S', r, r' the radii.

Then $$\frac{p}{p'}=\frac{PA\,.\,PB}{PA'\,.\,PB'}=\frac{Pa\,.\,Pb}{Pa'\,.\,Pb'}=\frac{PO^2-Oa^2}{PO'^2-O'a'^2}=\frac{PO^2-OX^2}{PO'^2-O'X^2}.$$

But $\dfrac{p}{p'}$ is also equal to $\dfrac{PO^2-r^2}{PO'^2-r'^2}$.

Whence $\dfrac{p}{p'}=\dfrac{OX^2-r^2}{O'X^2-r'^2}=\dfrac{x}{x'}$, i.e. P and X lie on a circle coaxal with S and S'.

If, then, it is desired to make the intersections of the line with the circles similar to those made by any four collinear points, say α, β, α', β', it is only necessary to set up a homothetic correspondence in which w and w', the mid-points of $\alpha\beta$ and $\alpha'\beta'$, correspond with O and O'. In this correspondence a will correspond to α, etc., the point P can be constructed, and the construction completed by drawing the circle on Pa as diameter and joining its intersection with S to the point P.

This note is suggested by proofs of the original construction by Mr. van der Heyden, and Prof. P. K. Kashikar, who also suggested the generalisation given here.

Similar proofs were received from E. Barton, G. W. Brewster, C. H. Hardingham, A. J. Hull, B. E. Lawrence, R. Sibson, L. J. Stroud, G. J. B. Westcott.

1316. *Volume of a tetrahedron.*

Let the vertices A, B, C, D be (x_r, y_r, z_r), $r = 1, 2, 3, 4$. If l, m, n are the direction cosines of the normal to the plane BCD and Δ the area of this triangle, we have by projection :

$$l\Delta = \text{area of triangle } (0, y_2, z_2)(0, y_3, z_3)(0, y_4, z_4)$$

$$= \tfrac{1}{2}\begin{vmatrix} y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \\ y_4, & z_4, & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}\alpha, \text{ say};$$

similarly $$m\Delta = \tfrac{1}{2}\begin{vmatrix} z_2, & x_2, & 1 \\ z_3, & x_3, & 1 \\ z_4, & x_4, & 1 \end{vmatrix} = \tfrac{1}{2}\beta,$$

and $$n\Delta = \tfrac{1}{2}\begin{vmatrix} x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \\ x_4, & y_4, & 1 \end{vmatrix} = \tfrac{1}{2}\gamma.$$

Hence $$\Delta = \tfrac{1}{2}\sqrt{(\alpha^2 + \beta^2 + \gamma^2)}.$$

Now the equation to the plane BCD is

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix} = 0,$$

that is $$x\alpha + y\beta + z\gamma = \begin{vmatrix} x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \\ x_4, & y_4, & z_4 \end{vmatrix}.$$

Denoting the perpendicular from A to this plane by p, the required volume

$$= \tfrac{1}{3}\, p\Delta$$

$$= \tfrac{1}{6}\, p\sqrt{(\alpha^2+\beta^2+\gamma^2)}$$

$$= \tfrac{1}{6}\left\{x_1\alpha + y_1\beta + z_1\gamma - \begin{vmatrix} x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \\ x_4, & y_4, & z_4 \end{vmatrix}\right\}$$

$$= \tfrac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}.$$

J. H. PEARCE.

1317. *The Circular points at infinity.*

The British Association Number of the *Gazette* (Nov. 1937) is " full of meat ". Not least interesting are the pages (343-52) on " Infinity " and " The Circular Points ". As an addendum to these, the following comments are offered :

(i) We tend to be too modest and apologetic in our use of the term " infinity " ; always inclined to suggest that it is *façon de parler*—convenient, but not essential ; e.g. " linguistic device . . . " (p. 346) ; " convenient expression . . . " (pp. 346, 350) ; " usually called . . . more customary nomenclature " (p. 349). (These quotations, not by way of criticism, but merely by way of illustrating a prevailing point of view.) I suggest that it is time we came out boldly with the affirmation that terms like " point at infinity ", " line at infinity ", . . . are essentially right ; and that it is the mathematically trained who hold the key to a conception, viz. " Infinity ", which is of the highest importance to human thought. The poets—especially the prophet-poets—have their own mode of intuitive apprehension of the relevant facts ; but there is no adequate " prose " of the subject except the mathematical, and it is time this were made more accessible to philosophic and other thinkers. This is too large a subject for further discussion in a note of this kind ; but it should be stated without qualification that the " points " of Euclidean Geometry are, in fact, of two kinds, viz. (1) the *ordinary points*, in which the term originates, and (2) the *points-at-infinity* (best so written, to accentuate the fact that while they are rightly called " points ", there is a certain " difference in kind " which is characteristic " paradox of infinity "). This " characteristic paradox " is that *difference in degree*, when it is *infinite difference*, exhibits features of *difference in kind* : a proposition which may be illustrated by Finite and Infinite Sequences and Series.

It is an unfortunate consequence of the " finite " human approach that the term " infinite " is negative in form. A truer order of ideas would be expressed by a *positive* term for what we call " infinite " (and the *corresponding negative* term for what we call " finite ").

(ii) Turning to the so-called " Circular Points at Infinity ", it is important to have in mind—in all such cases, where the " points ", in fact, " do not exist " (p. 350)—the essential relation (and distinction) between " Pure (Plane or Solid) Geometry " and the corresponding " Analytical Geometry ". Plane Geometry is, of course, *essentially two*-dimensional; but the corresponding Analytical Geometry—while " analytically " *two*-dimensional (in terms of the " correspondence ")—is, in its generality, " geometrically " *four*-dimensional (because " number ", when specified graphically, is two-dimensional). Thus, Plane Geometry is bound up with a four-dimensional geometry (in which it is included); and many of its features can be seen in their true perspective (of simplicity and beauty) only from the point of view of that relationship. But, unfortunately, our " space " intuitions are limited by the *three*-dimensional; and, so, we find ourselves in a kind of " cleft stick "—between a two-dimensional geometry of which we have adequate intuitive apprehension, but (so apprehending) realise to be theoretically part and parcel of a larger whole, and (on the other hand) that " larger whole "—of a four-dimensional geometry where our intuitive apprehension fails.

(This same difficulty besets us, in acute fashion, in the " Theory of Complex Functions "—where it would make all the difference if we could compound " the z-plane " and " the w-plane " into a clearly apprehended four-dimensional diagram.)

The so-called " Circular Points at Infinity " correspond to the fact that all the circles in a given plane are specifiable by second degree equations with common terms of that degree, viz. x^2+y^2 (Rect. Cart.)—the other coefficients g, f, c being " real " (and such that $g^2+f^2>c$). Actually, points-at-infinity on these circles (in the general type of case) " do not exist "; but in other analogous circumstances—such as that of x^2-y^2—such points-at-infinity (common to a family of hyperbolas) do exist. And, from the " analytical " point of view, it is all one question of the points-at-infinity, in *four*-dimensional Euclidean space, determined by an equation

$$A\,.\,X^2+2\,.\,H\,.\,X\,.\,Y+B\,.\,Y^2=0, \text{ where } X=x+\iota\,.\,x',\ Y=y+\iota\,.\,y'$$

—x, x', y, y' being all " real ", and A, B, H given coefficients (" real " or " unreal "). Such points-at-infinity always exist—a double infinity of them—but there need not be any of them in the xy-plane. In the particular case, here in question, they are determined by $Y=\pm\iota\,.\,X$, i.e. by $y=-x'$, $y'=x$ and by $y=x'$, $y'=-x$; and two are in the xy'-plane, two in the $x'y$-plane (in each case on the " bisector " lines)—none in the xy-plane or the $x'y'$-plane (vice versa for the case of X^2-Y^2).

The facts may be brought a stage nearer our geometrical intuitions by noting that when the " sectional " figure in the xy-plane is a circle with centre on the x-axis (i.e. when $f=0$), the " sectional " figure in the xy'-plane is the rectangular hyperbola specified by

$x^2 - y'^2 + 2g \,.\, x + c = 0$—in a *three*-dimensional diagram of the xyz'-space. In such a diagram—which may be drawn for the simplest case (viz. that of $X^2 + Y^2 = 1$) there can actually be *seen* points-at-infinity which satisfy the equation ; but they are not, of course, points of the circle.

(It will be observed that this Note is a positive amplification of the negative aspect of Professor McCrea's thesis—in part, " dotting *i*'s " and " crossing *t*'s ".) D. K. PICKEN.

1318. *Cubic envelopes.* (*Note* 1280.)

(The envelope of principal axes of conics touching four lines no doubt has its own literature ; the query first came to me about ten years ago from R. R. Webb of St. John's.)

I. The Jacobian (the name does not matter and might frighten the amateur) of three conic envelopes Σ, Σ', Σ'' is very easily identified as the envelope of the joins of the ∞^1 point-pairs in the net given by

$$\kappa\Sigma + \kappa'\Sigma' + \kappa''\Sigma'' = 0.$$

If specially $\Sigma'' \equiv \Omega$ is taken to be the circular points at infinity,the net of conics consists of the pencils of all the confocals of Σ and Σ'; the point-pairs are then the foci and the joins of the point-pairs are the principal axes. So the principal axes of the pencil $\lambda\Sigma + \lambda'\Sigma' = 0$ touching four fixed lines are given by the class cubic, which is the Jacobian of Σ, Σ', Ω. It must touch the line at infinity and the sides of the self-polar triangle of Σ and Σ'.

II. Specially consider Σ (general), Σ' (a circle centre P) and Ω. Since the cubic envelope is given by the Jacobian the conic can be replaced by any confocal and the circle by any concentric circle. The envelope now breaks up into the point P and a parabola ; either algebraically from the Jacobian or geometrically from the fact that the principal axes of the concentric circles pass through P.

Consider the four common tangents of the circle and Σ, where the radius of the circle tends to zero ; the contact points of the tangents from P to Σ are a point-pair of the net and so the parabola is the Chasles envelope of polar lines of P with respect to conics confocal with Σ. P. W. WOOD.

1319. *Hexagones inscriptibles remarquables.*

1. Soit un hexagone régulier $A_1A_2A_3A_4A_5A_6$, inscrit dans une circonférence de centre O et de rayon R, pour lequel le sens de circulation sur (O) est $A_1A_2A_3A_4A_5A_6$. Si l'on fait tourner le triangle équilatéral $A_1A_3A_5$, par exemple, d'un angle 2ϕ, que, pour fixer les idées, nous supposons compris entre 0 at $\pi/3$, autour du point O, les sommets du triangle $A_1A_3A_5$ prennent des positions A_1', A_3', A_5'. Les points A_1, A_1', A_3, A_3', A_5, A_5' sont les sommets d'un hexagone convexe inscrit dans la circonférence (O).

Dans ce polygone, on constate de suite que *les angles* A_1, A_1', A_3, A_3', A_5, A_5' *valent* 120°, que *les côtés opposés sont parallèles*, que *les diagonales* $A_1'A_3'$, $A_3'A_5'$, $A_5'A_1'$ *sont égales*, et que

(1) $$A_1A_1' = A_3A_3' = A_5A_5' = 2R\sin\phi,$$
$$A_1'A_3 = A_3'A_5 = A_5'A_1 = 2R\sin(60^\circ - \phi).$$

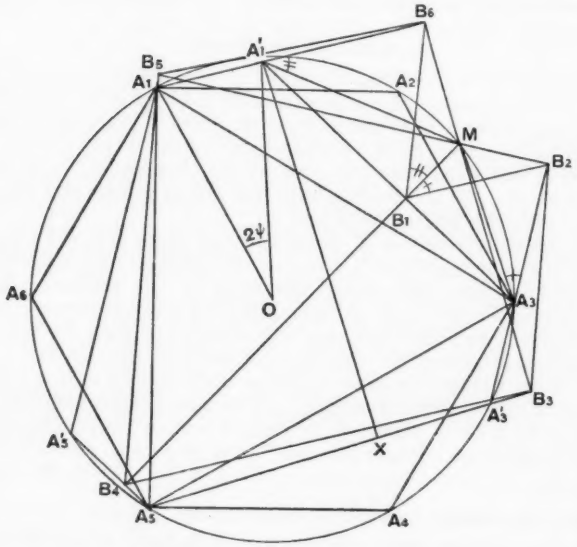
Il en résulte que *les triangles* $A_1A_3'A_3$, $A_3A_3'A_5$, $A_5A_5'A_1$ *sont égaux* et que *la somme de deux côtés consécutifs de l'hexagone* $A_1A_1'A_3A_3'A_5A_5'$ *est constante.*

L'aire S de cet hexagone est donc égale, en grandeur et en signe, à

(2) $$S = A_1A_3A_5 + A_1A_1'A_3 + A_3A_3'A_5 + A_5A_5'A_1$$
$$= A_1A_3A_5 + 3A_1'A_3'A_3.$$

Or, on a successivement, en vertu de (1), les relations d'aires

$$A_1A_3A_5 = \frac{3R^2\sqrt{3}}{4},$$



$$A_1A_1'A_3 = \tfrac{1}{2}A_1A_1' \,.\, A_1'A_3 \,.\, \sin A_1A_1'A_3 = \tfrac{1}{2}A_1A_1' \,.\, A_1'A_3 \,.\, \sin 120^\circ$$
$$= R^2\sqrt{3} \,.\, \sin\phi \,.\, \sin(60^\circ - \phi)$$
$$= \frac{R^2\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2} \,.\, \sin 2\phi - \sin^2\phi\right);$$

de sorte que

(3) $$S = \frac{3R^2\sqrt{3}}{4}(1 - 2\sin^2\phi + \sqrt{3} \,.\, \sin 2\phi).$$

Remarques. L'aire algébrique de l'hexagone $A_1A_1'A_3A_3'A_5A_5'$ devient nulle lorsque

$$1 - 2\sin^2\phi + \sqrt{3}\,.\,\sin 2\phi = 0,$$

c'est-à-dire, quand $\sin\phi$ vérifie l'équation du second degré

$$\sin^2\phi - 2\sqrt{3}\,.\,\sin\phi\cos\phi - \cos^2\phi = 0,$$

dont les racines

$$\sin\phi = (1 \pm 2)\cos\phi,$$

donnent, successivement,

$$\text{tang}\,\phi = 1 \pm 2,\ \sin\phi = -\frac{\sqrt{2}}{2},\ (\phi = 135°),\ \sin\phi = \frac{3\sqrt{10}}{10}.$$

Dans chacune de ces hypothèses, l'hexagone $A_1A_1'A_3A_3'A_5A_5'$ est *étoilé*.

2. Soient B_1, B_2, B_3, B_4, B_5, B_6 les projections orthogonales d'un point arbitraire M, que l'on peut supposer sur l'arc $A_1'A_3$ de la circonférence (O), sur les droites $A_1'A_3$, A_3A_3', $A_3'A_5$, A_5A_5', $A_5'A_1$, A_1A_1'.

Les égalités d'angles,

$$MB_1B_6 = MA_1'B_6,\quad MB_1B_2 = MA_3B_2, \ldots$$

donnent

$$(4)\qquad B_6B_1B_2 = \tfrac{1}{2}(A_1'OA_1 + A_3OA_1' + OA_3A_3') = \tfrac{1}{2}(A_3OA_1 + A_3'OA_3) = 60° + \phi_1, \ldots$$

et il en résulte que *les angles formés par les côtés consécutifs de l'hexagone podaire* $B_1B_2B_3B_4B_5B_6$ *du point* M, (ou leurs prolongements), *par rapport à l'hexagone* $A_1A_1'A_3A_3'A_5A_5'$, *sont égaux* ; en outre, *les côtés opposés de l'hexagone podaire sont parallèles.*

Soient a, b, c, d, e, f les longueurs des côtés B_1B_2, B_2B_3, B_3B_4, B_4B_5, B_5B_6 de l'hexagone. On a

$$a = \tfrac{1}{2}MA_2\sqrt{3},\quad b = \tfrac{1}{2}MA_3\sqrt{3},\quad c = \tfrac{1}{2}MA_4\sqrt{3},$$
$$d = \tfrac{1}{2}MA_5\sqrt{3},\quad e = \tfrac{1}{2}MA_6\sqrt{3},\quad f = \tfrac{1}{2}MA_1\sqrt{3},$$

et *le périmètre* P *de l'hexagone podaire est*

$$(5)\qquad P = \frac{\sqrt{3}}{2}\Sigma MA_1.$$

L'aire S de cet hexagone est la somme de deux parallèlogrammes de côtés respectifs b et c, e et f, et de mêmes angles $(60° + \phi)$ et $(120° - \phi)$; on a donc

$$S = (bc + ef)\,.\,\sin(60° + \phi) = \tfrac{3}{4}(MA_3\,.\,MA_4 + MA_1\,.\,MA_6)\,.\,\sin(60° + \phi).$$

Or,

$$MA_3\,.\,MA_4 = 2R\,.\,MB_3,\quad MA_1\,.\,MA_6 = 2R\,.\,MB_6\,;$$

donc

$$S = \frac{3R}{2}(MB_3 + MB_6) = \frac{3R}{2}\,.\,B_3B_6 = \frac{3R}{2}\,.\,A_1X,$$

A_1X étant la distance des côtés opposés A_1A_1', $A_3'A_5$.*

Comme

$$A_1X = A_1A_3 \sin A_1A_3A_4 = A_1A_3 \sin (60° + \phi)$$
$$= R\sqrt{3} \,.\, \sin (60° + \phi),$$

on a finalement

$$S = \frac{3R^2\sqrt{3}}{2} \,.\, \sin^2 (60° + \phi) = \frac{3R^2\sqrt{3}}{8} (\sin \phi + \sqrt{3} \,.\, \cos \phi)^2,$$

c'est-à-dire, après réductions,

$$S = \frac{3R^2\sqrt{3}}{8} (3 - 2 \sin^2 \phi + \sqrt{3} \,.\, \sin 2\phi)$$

$$= \frac{3R^2\sqrt{3}}{4} [1 + \tfrac{1}{2}(1 - 2 \sin^2 \phi + \sqrt{3} \,.\, \sin 2\phi)] \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$= \tfrac{1}{2}(A_1A_2A_3A_4A_5A_6 + A_1A_1'A_3A_3'A_5A_5'),$$

et *l'aire de l'hexagone podaire* $B_1B_2B_3B_4B_5B_6$ *est égale à la demi-somme des aires de l'hexagone régulier* $A_1A_2A_3A_4A_5A_6$ *et de l'hexagone* $A_1A_1'A_3A_3'A_5A_5'$, *inscrits dans la circonférence* (O).

Lorsque $\phi = 0$, l'hexagone $A_1A_1'A_3A_3'A_5A_5'$ coincide avec le triangle équilatéral $A_1A_3A_5$ et l'hexagone podaire du point M est formé de la droite de SIMSON de M par rapport à $A_1A_3A_5$ et du triangle podaire $B_2B_4B_6$ de M pour le *triangle tangentiel* en A_1, A_3, A_5. Dans ce cas,

$$S = B_2B_4B_6 = \tfrac{1}{2}\left[\frac{3R^2\sqrt{3}}{2}(1 + \tfrac{1}{2})\right] = \frac{9R^2\sqrt{3}}{8}.$$

Si $\phi = 30°$, les deux hexagones $A_1A_2A_3A_4A_5A_6$ et $A_1A_1'A_3A_3'A_5A_5'$ coïncident, et l'on retrouve ces propriétés relatives à l'hexagone régulier :

l'aire de l'hexagone podaire d'un point quelconque de la circonférence circonscrite, par rapport à un hexagone régulier, est égale à celle de l'hexagone régulier.† *Les côtés consécutifs de l'hexagone podaire sont rectangulaires.*‡

Remarques. L'hexagone $A_1A_1'A_3A_3'A_5A_5'$ possède d'autres propriétés qui rappellent celles de l'hexagone régulier.

Ainsi, *le lieu d'un point M du plan tel que l'aire,* (ou la somme des carrés des côtés), *de son hexagone podaire, par rapport à*

$$A_1A_1'A_3A_3'A_5A_5',$$

soit égale à une constante k^2, *est une circonférence concentrique à la circonférence* (O) *circonscrite, et réciproquement.*

En effet, dans les équations

$$\Sigma \overline{MA}_1^2 \sin 2A_1 = k^2, \text{ (ou } \Sigma \overline{MA}_1^2 \sin^2 A_1 = k^2),$$

* Les côtés opposés de l'hexagone $A_1A_1'A_3A_3'A_5A_5'$ sont équidistants.

† V. Thébault, *Mathesis*, 1926, 372.

‡ A. Boutin, *Intermédiaire des mathématiciens*, 1923, 49.

des lieux cherchés, il suffit d'observer que les angles A_1, A_2, A_3, A_4, A_5, A_6 de l'hexagone considéré sont égaux ; de sorte que le centre des moyennes distances des sommets A_1, A_2, A_3, A_4, A_5, A_6, chargés respectivement de masses égales $\sin 2A_1 = \sin 2A_2 = \ldots = \sin 2A_6$ (ou $\sin^2 A_1 = \sin^2 A_2 = \ldots = \sin^2 A_6$), coïncide avec le centre O de la circonférence circonscrite.

V. THÉBAULT.

1320. *The addition formulae.*

Is the following proof, which seems to appeal to boys, well known?

In any triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Thus
$$\frac{b\cos C + c\cos B}{\sin(B+C)} = \frac{b\cos C}{\sin B\cos C} = \frac{c\cos B}{\cos B\sin C}$$
$$= \frac{b\cos C + c\cos B}{\sin B\cos C + \cos B\sin C}.$$

Therefore
$$\sin(B+C) = \sin B\cos C + \cos B\sin C.$$

E. P. LEWIS.

1321. *A note on mathematical expectation.*

In books on probability and statistics the various ways in which probability can be defined are mentioned. The relative frequency definition is often used in statistics. However, the reason why the mathematical expectation is generally not a whole number in the case of a sample of finite size when in the actual case the relevant number has to be a positive integer, is often not explained. An explanation is offered here on the basis of the relative frequency definition.

We assume that we are given an infinite hypothetical population of objects. We consider a random sample of n objects—i.e. of *size* n. Let m be the number of objects in the sample possessing a prescribed property. m is the frequency and $\frac{m}{n}$ the relative frequency of the occurrence of the property in the sample ; let $f_n = \frac{m}{n}$.

We suppose now that random samples of varying sizes are drawn from the population : the values of m and n will vary with the samples. There will be a set of relative frequencies

$$f_1, f_2, \ldots, f_n, \ldots,$$

associated with the set of samples, of the occurrence of the prescribed property.

It is now assumed that if $n \to \infty$, f_n also tends to a unique limit. Let

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \left(\frac{m}{n}\right) = p.$$

p is called the statistical probability of an object in the population, selected at random, having the prescribed property. It is clear that $0 \leqslant p \leqslant 1$.

Consider now a random sample of size N; let M be the number of objects in it having the prescribed property.

pN is called the "expected" or "probable" number (mathematical expectation of the number) of objects in the sample possessing the property. It is necessary to observe that pN may not, and will generally not, be equal to M for the following reason:

$$M = \frac{M}{N}\, N = f_N N.$$

As $p = \lim_{n\to\infty} f_n$, f_N may not, and will generally not, equal p.

Consequently pN, the "expected" or the "probable" number of objects having the prescribed property, will not generally equal the actual number of objects in the sample possessing the property. Moreover, for the same reason pN may not even be a whole number.

It will be useful to give two examples illustrative of the foregoing explanation.

(1) We take the common example of tossing a coin. The *a priori* probability of obtaining a head or a tail is $\frac{1}{2}$. It is assumed that the *a priori* probability is equal to the statistical probability.

Each toss is an object of the infinite hypothetical population, relevant to the present question.

Then,

for $2N$ tosses, the "expected" number of heads will be

$$\tfrac{1}{2} \times 2N = N,$$

and

for $2N+1$ tosses, the "expected" number of heads will be

$$\tfrac{1}{2} \times (2N+1) = N + \tfrac{1}{2},$$

N being a whole number. It is needless to say that when an experiment is performed, the number of heads actually obtained is always an integer.

(2) Lastly we consider an example from the kinetic theory of gases and stellar dynamics, as statistical methods are used in these subjects.

Let u, v, w be the components of the velocity of a $\left\{\begin{matrix}\text{molecule}\\ \text{star}\end{matrix}\right\}$ along three mutually perpendicular directions.

The Maxwellian velocity distribution function is given by

$$f(u, v, w) = \frac{1}{(\sqrt{2\pi})^3} e^{-\frac{1}{2}(u^2+v^2+w^2)}.$$

(There is no loss of generality. The units are assumed to be suitably chosen in each case.)

According to the usual definition, the probability that the velocity of a $\left\{\begin{matrix}\text{molecule}\\ \text{star}\end{matrix}\right\}$, selected at random, has its components between the limits u and $u+du$, v and $v+dv$, and w and $w+dw$ is

$$f(u, v, w)du\,dv\,dw\,;$$

and the probability of the velocity of the $\left\{\begin{matrix}\text{molecule}\\ \text{star}\end{matrix}\right\}$ referred to having its components between the limits U_1 and U_2, V_1 and V_2, and W_1 and W_2 is

$$\iiint f(u, v, w)du\,dv\,dw,$$

where the integral is taken over the volume of the rectangular parallelopiped determined by the points (U_1, V_1, W_1), (U_1, V_1, W_2), ... , (U_2, V_2, W_2) in the uvw space.

The probability of the u-component of the velocity of a $\left\{\begin{matrix}\text{molecule}\\ \text{star}\end{matrix}\right\}$ being between the limits U_1 and U_2 is obtained by putting $V_1 = -\infty$, $V_2 = +\infty$ and $W_1 = -\infty$, $W_2 = +\infty$ in the above volume integral and is equal to

$$\int_{u=U_1}^{U_2}\int_{v=-\infty}^{\infty}\int_{w=-\infty}^{\infty} f(u, v, w)\,du\,dv\,dw$$

$$= \frac{1}{(\sqrt{2\pi})^3}\int_{u=U_1}^{U_2}\int_{v=-\infty}^{\infty}\int_{w=-\infty}^{\infty} e^{-\frac{1}{2}(u^2+v^2+w^2)}\,du\,dv\,dw$$

$$= \frac{1}{\sqrt{2\pi}}\int_{u=U_1}^{U_2} e^{-\frac{1}{2}u^2}du, \text{ since } \int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2}dv = \int_{-\infty}^{\infty} e^{-\frac{1}{2}w^2}dw = \sqrt{2\pi}.$$

Suppose we have a volume of gas having N_1 molecules and a sample of N_2 stars.

The " expected " number of molecules whose u-components lie between U_1 and U_2

$$= \frac{N_1}{\sqrt{2\pi}}\int_{U_1}^{U_2} e^{-\frac{1}{2}u^2}du.$$

The " expected " number of stars in the sample whose u-components lie between U_1 and U_2

$$= \frac{N_2}{\sqrt{2\pi}}\int_{U_1}^{U_2} e^{-\frac{1}{2}u^2}du.$$

In samples of gases, the number N_1 is very large; and also, provided that U_1 and U_2 are large enough,

$$\frac{1}{\sqrt{2\pi}}\int_{U_1}^{U_2} e^{-\frac{1}{2}u^2}du$$

would be very small.

Consequently the "expected" number can become less than unity. In any case, the "expected" number will, in many cases, not be a whole number.

Similarly in the case of samples of stars, of size large or small, the "expected" number can become less than unity; and it will, in many cases, not be a whole number. C. G. PENDSE.

1322. *The sum of a harmonical progression.*

1. *Introduction.*

We shall give here an expression for the sum of a harmonical progression by means of the Fractional Calculus.

The λth integral of a uniform function $f(z)$ along a simple curve l is defined by *

$$D^{-\lambda}(l_a)f(z) = \frac{D^\gamma}{\Gamma(\lambda+\gamma)}\int_a^z (z-t)^{\lambda+\gamma-1} f(t)dt,$$

where γ is the least integer greater than or equal to zero such that $R(\lambda)+\gamma > 0$; D stands for $\frac{d}{dz}$, and the integration and differentiation are along l.

Now let C be a closed contour formed by two simple curves l and L joining points a and z_0, which do not cross each other. Keeping C and z_0 fixed, let the length of L tend to zero *after* $D^{-\lambda}(l_a)f(z_0)$ has been evaluated, by letting a move along C until it coincides with z_0. Then l becomes C, and the value of $D^{-\lambda}(l_a)f(z_0)$, when l has become C by this process, is denoted by † $D^{-\lambda}(C)f(z_0)$.

Consider the series

$$S(z_0, k) = \sum_{m=1}^{n} A_m (z_0 - b_m)^k,$$

where the numbers A_m, b_m and k, which may be real or complex, are independent of z_0.

Let $f(z)$ be analytic on and within a contour C except for poles inside C at the points $b_1, b_2, \ldots b_n$, at which the principal parts of $f(z)$ are

$$\frac{A_1}{(z-b_1)^\sigma}, \frac{A_2}{(z-b_2)^\sigma}, \cdots \frac{A_n}{(z-b_n)^\sigma}$$

* Fabian, *Math. Gazette*, vol. xx, No. 240 (1936), pp. 249-53.

† Fabian, *Phil. Mag.*, ser. 7, vol. 21 (1936), pp. 275, 276.

respectively. Then we have, as shown in a previous paper,*

$$S(z_0, k) = \frac{(-1)^{\sigma-1}}{2\pi i}\Gamma(k+1)\,.\,(\sigma-1)\,!\,D^{-k-\sigma-\delta}(C)f^{(\delta)}(z_0),$$

where δ is the least integer greater than or equal to zero such that $R(k)+\sigma+\delta>0$; and $\Gamma(k+1)D^{-k-\sigma-\delta}(C)f^{(\delta)}(z_0)$ stands for

$$\lim_{\mu\to k}\Gamma(\mu+1)D^{-\mu-\sigma-\delta}(C)f^{(\delta)}(z_0)$$

when k is a negative integer.

2. *Harmonical Progressions.*

The results of the last paragraph will now be used to obtain an expression for the sum of a harmonical progression.

Consider the harmonical progression

$$S=\sum_{m=0}^{n}\frac{1}{\beta+mb},$$

n being a finite number.

Let $\varphi(z)$ be a Weierstrass's elliptic function which has b for one of its periods. Our result in the last paragraph gives us :

$$S=-\frac{1}{2\pi i}\lim_{\mu\to-1}\Gamma(\mu+1)D^{-\mu-2}(C)\varphi(\beta),$$

where $\varphi(z)$ is analytic on† and within the contour C except for poles inside C at the points $0,\ -b,\ -2b,\ \ldots\ -nb$.

Similarly, S can be expressed in terms of Jacobian elliptic functions.

W. FABIAN.

1323. $2=4$: *a new proof.*

It is known that the process of repeated exponentiation $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ (where each x is raised to the power of all that follows it) converges to a definite limit $f(x)$ if x lies between e^{-e} and $e^{1/e}$, approximately ·066 and 1·44. (See, e.g., D. F. Barrow, *Amer. Math. Monthly*, **43** (1936), 150-160.) The value $x=\sqrt{2}$ lies within this range; and for this value of x, by repeated substitution of x^2 for 2 in the index,

$$2=x^2=x^{x^2}=x^{x^{x^2}}=\ldots=x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}=f(\sqrt{2}).$$

Similarly, for the same value of x,

$$4=x^4=x^{x^4}=x^{x^{x^4}}=\ldots=x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}=f(\sqrt{2}),$$

which proves the proposition.

I. M. H. ETHERINGTON.

* Fabian, *Phil. Mag.*, ser. 7, vol. 21 (1936), p. 278.

† For this to be possible, β must not be a multiple of $\pm b$.

1324. *An adventure with limits.*

It appears not to be generally known (although the method is to be found in some textbooks) that the area under the curve $y=x^n$ is easy to deduce from first principles as the limit of a sum if the base is divided into segments not equal, but in geometrical progression.

Thus to find the area under $y=x^2$ from $x=0$ to $x=a$, divide the base at P_1, P_2, P_3, ... , where $OP_1=ar$, $OP_2=ar^2$, $OP_3=ar^3$, ... Then the area is the limit as $r\to 1$ of

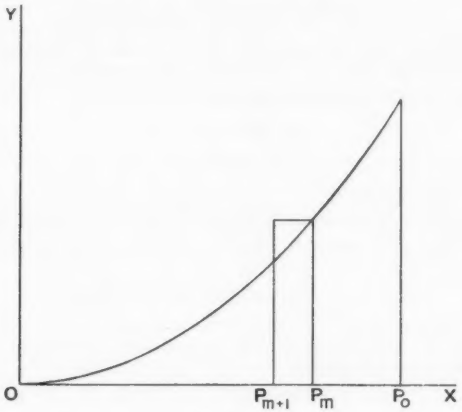


FIG. 1.

$$\sum_{m=0}^{\infty} a^2r^{2m}(ar^m-ar^{m+1})=a^3(1-r)/(1-r^3)$$
$$=a^3/(1+r+r^2),$$

which is easily seen to be $a^3/3$. For any rational index (other than -1!) the result can be obtained with almost equal ease.

The author of this note has not, however, seen the method applied to $y=1/x$. As a particular case, let us take the area under this curve from $x=1$ to $x=2$. Let $OP_1=r$, $OP_2=r^2$, ... , with n segments, so that $r^n=2$. Then (see Fig. 2)

$$P_mQ_m=y_m=r^{-m},\quad P_mP_{m+1}=r^{m+1}-r^m=r^m(r-1).$$

The area of the " outer " rectangle $P_mQ_mR_{m+1}P_{m+1}$ is then

$$r^{-m}\,.\,r^m(r-1)=(r-1),$$

so that the total area under the curve is the limit as $r\to 1$ of

$$n(r-1)=n(2^{1/n}-1)=(2^x-1)/x,$$

where $x=1/n$, and the limit as $x\to 0$ is now to be sought.

Now

$$2^x = (1+1)^x$$
$$= 1 + x + \frac{x(x-1)}{2!} + \frac{x(x-1)(x-2)}{3!} + \ldots$$

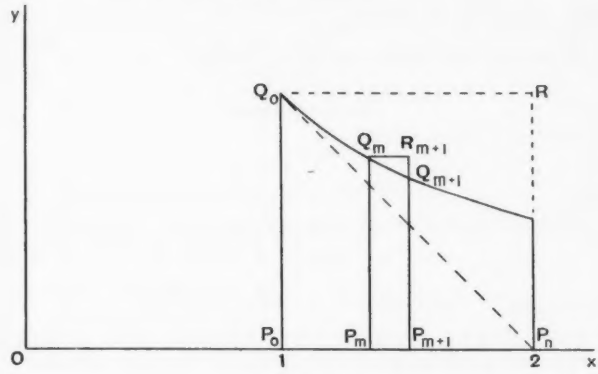


FIG. 2.

so that

$$\frac{2^x - 1}{x} = 1 - \frac{1-x}{2} + \frac{(1-x)(1-x/2)}{3}, -\ldots,$$

and the limit appears as the sum of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$$

(The limiting process can be justified, since the binomial series converges for the values of x in question.)

The usual method of grouping terms shows that the sum of this series lies between 1 and 1/2. Geometrically these limits are represented by the rectangle $P_0Q_0RP_n$ and the triangle $P_0Q_0P_n$, Q_0P_n coinciding with the tangent at Q_0. The actual numerical value of the area is only obtainable from this series with prohibitive labour.

If, as is usual, calculus is introduced before the exponential and logarithmic series, and if, following history, the problems of the integral calculus are examined before the method of solution via the differential calculus is shown, the above may provide an intriguing conundrum for the pupils to ponder over.

It is possible, too, by an easy extension, to obtain the result in a form which permits calculation. To do this, we have merely to write $2 = (1+1/3)/(1-1/3)$, and then evaluate by the same method the limit as $x \to 0$ of

$$\frac{(1+1/3)^x - (1-1/3)^x}{x(1-1/3)^x},$$

which easily yields the rapidly convergent series

$$2\left(\frac{1}{3}+\frac{1}{3\,.\,3^3}+\frac{1}{5\,.\,3^5}+\ldots\right).$$

W. G. B.

1325. *Three curious problems.*

1. Show that the infinite continued fraction

$$\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\ldots}}}$$

where

$$a_n=\cfrac{1}{n+\cfrac{1}{n+\cfrac{1}{n+\ldots}}}$$

is convergent.

2. Prove that for any prime number p there exists at least one primitive root g and at least one number $k=k(p,g)$ such that

$$g^K\equiv K\,(\mathrm{mod}\ p).$$

3. Prove that the set of all the permutations of the natural series of numbers $(1, 2, 3, \ldots)$ has the power of continuum.

G. Highdooc.

1326. *The general solution of* $ax+by=c$.

Since

$$x+\frac{b}{a}\,y=\frac{c}{a},$$

if x and y are integers then $(by-c)/a$ is an integer, and so is

$$(kby-kc)/a,$$

where we take $kb\equiv 1\ (\mathrm{mod}\ a)$.

Let $$kb=la+1,\quad kc=ma+d.$$

Then $(y-d)/a$ is an integer, say p.

Thus $$y=ap+d. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Substitution of this value now gives

$$\begin{aligned} x&=-b(ap+d)/a+c/a\\ &=-bp-\{b(kc-ma)\}/a+c/a\\ &=-bp-\{(la+1)c-mba\}/a+c/a\\ &=-bp-lc+bm. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}\end{aligned}$$

The only point to be determined is the value of k and the rest follows mechanically.

Example. $5x+3y=19$.

Here $a=5,\quad b=3,\quad c=19$;

$$kb=3k\equiv 1 \pmod 5.$$

Thus we take $k=2$ and $l=1$.

$$kc=38=5m+d.$$

Hence $m=7,\quad d=3$.

Thus
$$y=5p+3,$$
$$x=-3p-19+21$$
$$=2-3p.$$

N. K. NARASIMHA MURTHY.

1327. *A point in the teaching of integration.*

The following " catch " may amuse the readers of the *Gazette* and perhaps be of use in teaching.

$$\int_{2\cos A}^{1}\frac{adx}{\sqrt{(1-a^2x^2)}}=\text{arc sin}(a)-\text{arc sin}(2a\cos A).$$

Put $a=\sin A$ and the right-hand side becomes

$$\text{arc sin}(\sin A)-\text{arc sin}(\sin 2A)=A-2A$$
$$=-A.$$

Put $A=\frac{1}{3}\pi$; the limits of the integral then coincide and the integral vanishes, giving

$$0=-\tfrac{1}{3}\pi.$$

The solution, of course, is that if $\frac{1}{4}\pi<A<\frac{1}{2}\pi$ the principal value of arc sin (sin $2A$) is not $2A$ but $\pi-2A$, but a beginner in the calculus might be puzzled by the difficulty and instructed by the explanation.

H. WALLIS CHAPMAN.

1328. *The complete pentagon.*

I should be grateful to anyone who can supply me with references to properties of the complete pentagon, such as these which I have worked out for myself:

(i) the orthocentre of the ten triangles formed lie four by four on five lines, and so on;

(ii) the joins of the middle points of the diagonals of the five quadrilaterals meet in the centre of the conic touching the five lines (by Plücker's theorem);

(iii) the " Wallace " points of the five quadrilaterals are concyclic.

N. M. GIBBINS.

1209. NAPOLEON UNDER THE DIRECTORY.

At the Institute, he was to be seen always seated between Lagrange and Laplace, apparently wholly occupied with the abstract sciences.—Alison, *History of Europe*, IV, p. 167.

REVIEWS.

Methoden und Probleme der dynamischen Meteorologie. By H. ERTEL. Pp. 122. RM. 14. 1938. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band V, Heft 3. (Julius Springer, Berlin)

In this little book of three chapters the author has compressed into two short chapters such portions of thermodynamics, radiation theory, hydrodynamics and general dynamics of the atmosphere as are required for the discussion of atmospheric problems, while the third chapter is devoted to the discussion of certain special problems of the dynamics of the atmosphere. The first two chapters are too much compressed to be readily studied by beginners in the subject of Meteorology, but to the advanced student they will appeal as a most useful résumé of the subjects covered, the references to original papers being particularly well-selected. The reader will have to be familiar with the tensor notation if he is to follow the whole of these chapters, though most of the important results are given in cartesian co-ordinates.

The special problems in the dynamics of the atmosphere treated in the third chapter include the questions of stability and instability, the explanation of the mean distribution of temperature in the vertical, quasi-static changes of state, the variation of wind with height in the surface layers, stationary surfaces of discontinuity in the atmosphere, the equations of perturbation, and non-stationary motion.

Such a book cannot fail to be of the greatest use to the serious worker in Meteorology, in that it provides in a readily usable form most of the mathematical equipment required for the discussion of meteorological problems, whose solution is, in many cases, still lacking. The difficulty which always arises in attempts to solve problems in atmospheric motion is that the full equations are so complex as to be hopelessly intractable, so that some simplification is necessary. The only sound plan of simplification is of necessity based on an appreciation of the physical principles involved in the process under discussion. It is in the lack of emphasis on the physical principles involved in atmospheric processes that the present book appears to the reviewer to be most lacking. This is perhaps most clearly exemplified in the last few pages of the book, where the motion of air through a field of pressure is discussed on the basis of an assumption made by Möller and Sieber, that the velocity of the air lags behind the velocity appropriate to its instantaneous position, the velocity at time t being the velocity appropriate to the position of the air at time $t-\tau$. This artificial assumption has the virtue that it leads to tractable equations, but physically it appears to be an unwarranted assumption to make, and it does not appear to be a mathematical necessity.

Dr. Ertel has performed a useful task in producing this little book, which should be regarded as a supplement to the usual textbooks, rather than as an attempt to produce another textbook. D. B.

Isaac Newton. 1642-1727. By J. W. N. SULLIVAN. Pp. xx, 275. 8s. 6d. 1938. (Macmillan)

The death of Sullivan in the summer of last year deprived the history of science of one of its most gifted exponents. He was not a professional scientist, but his freedom from immersion in technical details helped to give that clearness of insight which is one of the characteristics of his writings—a characteristic which contributed greatly to his skill in scientific popularization. His last work, finished just before his death, was the biography under review, in which he successfully attempts to relate the work and attainments of Newton to his personality and temperament.

The recent life by Professor More, with its examination of original sources and its careful documentation, cleared away many of the myths and false ideas which had concealed the true character of Newton. The present book could not, and is not intended to, replace the earlier one, but supplements it by presenting more briefly the salient features of his life and their psychological background. Of these features, the most striking is undoubtedly his complete indifference to the opinion and estimation of his contemporaries, save in those cases where attacks seemed to be made on his honesty. This was partly inherent, but was in part acquired as a means of defence against the demands made on his time and patience by others. In his early writings he shows an enthusiasm for his discoveries that he feels will be reciprocated by others ; but this quickly disappears, to be replaced by a cold objectivity which in time becomes a reluctance to publish at all, except when compelled to do so by the pressure of circumstances. Another baffling feature of his life is the ease with which he abandons for long periods the work at which he is pre-eminent, although, when occasion requires, he shows his mastery is unimpaired. The solution of the enigma lies, as Sullivan states in ending his book, in the fact " that Newton was a genius of the first order at something he did not consider to be of the first importance ". He did not regard the facts revealed by science as capable of reaching what he regarded as truth, and he thought a life devoted to science was largely wasted. Hence followed his devotion to theology, to him the study of divine revelation, which he pursued with a continuity and tenacity such as were only engaged for short periods at a time in his scientific pursuits.

As mentioned before he only defended himself when his honour was attacked. Then, his usual generosity vanished, and all his magnificent powers were devoted to clearing his name of the aspersions cast upon it. At other times, he was ever ready to acknowledge the merits of others' work, and always prepared to help younger mathematicians who came to him for aid.

These remarks will show the trend of Sullivan's book. It is delightfully written, and can be warmly recommended to all, whether scientists or not, who are interested in the character of one of the outstanding figures of human history. C. W. G.

A History of Science, Technology, and Philosophy in the 16th and 17th Centuries. By A. WOLF. Pp. xxvii, 692. 25s. 1935. (Allen and Unwin)

Professor Wolf has undertaken a task which is almost forbidding in its magnitude, in proposing to write a history of science from ancient to modern times. It is a task often attempted by the encyclopaedists of former centuries, but the enormously extended scope of each science, and the greater knowledge and accuracy expected from a modern writer, render comparison with earlier adventurers useless. The present volume covers the sixteenth and seventeenth centuries, and it may be said at once that Professor Wolf justifies his seeming audacity by the high level of success achieved.

No period is of greater interest in scientific history. During it, mathematical and physical science was established on its permanent foundations by the work of Galileo, Kepler, Newton and their contemporaries, and other sciences, though finding it much more difficult to discard their medieval shackles, yet found means to make the first stumbling steps towards a dimly seen goal. All this and much more, is described in this fascinating volume, and although the mathematician may first read those chapters most germane to his subject, he will soon find himself turning the pages to discover unknown facts about early medicine, or building, or precursors of the steam engine.

The first third of the book describes the revolution started by Copernicus,

and its results in the domain of astronomy, mathematics, and physics. The awakening of accurate scientific interest first showed itself in these subjects, and the names of Copernicus and Galileo are immortal, as those of leaders in the revolt against prejudice, and as those of fighters for freedom of thought. A full account is given of their work, and the bitter opposition against which they struggled. The success of their fight was more rapid than might have been anticipated, and the position they had won was consolidated by the scientific academies which were established during this period, and of which an interesting account is given. Kepler's work continued the emancipation of thought, and it was enormously aided by the familiar scientific instruments, most of which date from this epoch. The development from their work to the embracing generalizations of Newton and the mathematical discoveries made during the time are too familiar to readers of the *Gazette* to require recapitulation. It is sufficient to say that the account given is an authoritative one, to which objection can, at most, only be made on minor points.

To the other sections of the work, the mathematical reviewer comes as an enquiring outsider. There are chapters dealing with the biological sciences, medicine, and all branches of technology, with final chapters on psychology, the social sciences, and philosophy. The last named, in a masterly summary, presents the background of thought into which the new discoveries fitted, and from which they required enormous adjustments.

Opening the book at random, one is immediately engrossed by the account of some curious speculation, construction, or instrument which has played its part in the steady onward march of scientific discovery. Nearly every page has its illustration, many of which are reproduced from contemporary works, and which frequently enable one to grasp the construction, use, and theory of apparatus in a manner otherwise impossible. The further one reads, the more one is impressed by the wide reading and profound erudition of the author.

The printing is excellent, and the appearance of the volume worthy of the place it will occupy on the shelves of those fortunate enough to possess it.

C. W. G.

The General Field Theory of Schouten and van Dantzig. By N. G. SHABDE. Pp. ii, 55. N.p. 1938. Lucknow University Studies, X. (University of Lucknow)

In this series of three lectures Professor Shabde gives an exceedingly clear account of the " unified " field-theory of Schouten and van Dantzig and of his own contributions to it. As is very well known, Einstein's general relativity gives a complete *geometrization* of the physics of gravitational phenomena, but leaves no room within its own scheme for electromagnetic phenomena. If it is possible to extend the geometrization to include these phenomena, it must be through the use of some geometry more general than the Riemannian geometry used in general relativity. In 1928 Einstein himself initiated a series of attempts to do this, nearly all of which have depended on some introduction of a fifth coordinate in addition to the four used in general relativity. At first sight there appear to be too many such possible generalizations of Riemannian geometry to choose from. But if a concise classification of the possibilities can be given the selection of an appropriate one may become feasible. This really supplies one of the main interests of Professor Shabde's work, since he shows that other proposed unified field-theories, namely those of Einstein and Mayer and of Veblen and Hoffmann, are particular cases of that of Schouten and van Dantzig, thus providing the desired classification as far as these three principal unified theories are concerned.

The theory of Schouten and van Dantzig is a form of " projective " relativity

and the first Lecture gives a summary of the appropriate projective differential geometry. In the second Lecture the field equations are derived from a variational principle. In the third, identities between the field equations, and the connection between general relativity and this theory, are discussed. The mathematical treatment is admirably complete. One regrets, however, that the author considers it outside his province to give any explanation of what is gained *physically* from the unification, i.e. whether any observable phenomena involving both gravitation and electromagnetism can be treated by this theory better than by previous methods, or whether any new phenomena of this sort are predicted by the theory. W. H. McC.

A New Algebra for Schools. Part III (Shorter Version). By C. V. DURELL. Pp. iv, 329–435. With answers, 2s. 1938. (Bell)

The subject matter is taken from Part III of the author's *New Algebra* (see *Gazette*, July 1932) and is treated along the lines of his *School Certificate Algebra* (*Gazette*, Dec. 1937). It represents a lowest common multiple of the requirements of the various examination boards in respect of their " Elementary " syllabuses for School Certificate.

A introductory chapter " Further Processes " covers elementary quadratic theory, remainder theorem, and square root. Indices, surds, and logarithmic theory and computation, the latter assuming a base ten, provide the material of a second, while a comprehensive treatment of ratio, proportion, variation, A.P. and G.P. completes the volume.

The examples are numerous and varied, and distinguishing marks add to their value. Bracketed numbers provide the " Extra practice " of the earlier volume and bright boys will no doubt be anxious to add their private marks to the author's asterisks.

Revision exercises, test papers, and tables of logarithms, find their place and the result is a handy volume which should meet a definite need. E. L.

Primary Mental Abilities. By L. L. THURSTONE. Pp. ix, 121. 9s. 1938. Psychometric Monographs, Number I. (Published for the Psychometric Society in U.S.A. from the University of Chicago Press, in Great Britain and Ireland from Cambridge University Press.)

For several years students of the application of factor methods to the results of psychological testing have awaited the results of the grand-scale experiment carried out by Professor L. L. Thurstone and his assistants at the University of Chicago. A preliminary account of their work appeared in *Psychometrica* in September 1936, and now the complete report is available as Psychometric Monograph Number I.

Ever since Spearman's 1904 paper where " g " was first put forward as a universal primary mental ability there has been a small yet increasing band of investigators who have attempted to isolate and measure other fundamental human abilities. Some evidence has been advanced for the existence of verbal ability, spatial ability, number ability and many others, but there are those who consider this evidence inadequate.

By assuming the scores obtained in tests to be linear combinations of uncorrelated variables, it is a simple matter to resolve a table of intercorrelations of test scores into a matrix of weightings of the uncorrelated variables. There is an infinity of solutions; the difficulty arises when we try to select one solution which has the most plausible psychological interpretation. The experimental obstacles are legion and it is only by large-scale experiments carefully carried out that we can hope to get data which will permit of resolution in such a way as to give reliable results with psychological meaning. Professor Thurstone is one of the most careful experimenters in America and this volume,

which represents a phenomenal amount of work, shows on every page evidence of his scientific care for details.

Sixty tests (which may be obtained in a separate volume) were administered to 240 subjects, each of whom spent 15 hours in the testing room. The tests may be roughly classified into verbal tests, spatial tests, number tests, reasoning tests and memory tests. Correlations between results were obtained and the table of intercorrelations resolved into factors by the Thurstone Centre of Gravity Method. Special criteria were adopted for finding the most significant solution. In this way it was found that the scores on all the tests could be attributed to the effects of twelve factors. Seven of these have been identified with some confidence ; further work is being carried out with the remainder. The factors thus identified are (1) spatial ability, (2) perception, (3) number ability, (4 and 5) verbal abilities, (6) memory, (7) inductive reasoning.

Several interesting points regarding these may be noted :

Although the battery of tests included tests requiring manipulation both in flat space and in three dimensions, there was evidence of only one spatial factor in both forms of test.

The number ability was limited to the four fundamental operations and could in no wide sense be termed mathematical ability.

Two verbal factors were evident : one which related to fluency in the use of single words, and the other ability to use ideas verbally in sentence form.

There is evidence that inductive reasoning requires mental processes distinct from deductive reasoning.

Thurstone has not isolated Spearman's " g ". It cannot be assumed however that his results are evidence in disproof of " g "; it is quite possible that this factor may be present in the data as a function of the unresolved variance.

The book describes the most valuable educational experiment of its kind hitherto carried out. All students of factor methods should read it, and anyone wishing to become acquainted with this new technique of educational research will find here an interesting introduction.

A valuable beginning has been made at Chicago but we are left with the feeling that much remains to be consolidated before results of this kind can be translated into terms of significance for classroom practice. We are assured by Thurstone that this work is in progress. C. E. S.

The Psychology and Teaching of Arithmetic. By H. G. WHEAT. Pp. x, 591. 8s. 6d. 1936. (D. C. Heath, Boston ; Harrap)

This book is of interest to English readers for its careful historical account of the development of the number system and its emphasis upon the relative recency of our adoption of the Hindu-Arabic notation. It traces the stages by which the ancient art of reckoning gave place to the modern science of arithmetic, and it describes the steps by which pupils may be led to think in terms of groups of tens and to utilize the unified body of ideas which has been formulated as a convenient means of dealing with quantitative situations.

Some knowledge of contemporary American textbooks is, however, necessary before one can appreciate the need for so much insistence on the importance of the pupil's realization of the significance of the ideas of position and of size. Textbook makers in this country have not passed through the phases of extreme elimination of subject matter on the plea of social inutility, of extreme conviction of the value of drill unenlightened by explanation, of extreme absorption in the planning of activities by which arithmetical learning may be merely incidental. Consequently, this author's 583 pages of detailed attacks on these types of approach seem to us leisurely to the point of tediousness. His position is similar to that of W. A. Brownell and others in the Tenth Year-

book of the National Council of Teachers of Mathematics,* but it is expressed with much less brevity and is consequently less likely to influence current practice.

Readers in search of discussions of methods of teaching will, however, be interested in his advocacy of subtraction before addition, division before multiplication, in his postponement of short division and remainders, and in his efforts to clarify the relationships of vulgar fractions, decimals and percentage. They will also be stimulated by his attempt to restore the use of problems to a balanced position as neither a mere motive for learning nor a mere opportunity for applying what has been learned, but a means of presenting number ideas in a variety of familiar situations in order that the ideas may be clarified, enlarged and recognized. They will remember his analysis of two step problems into ten possible combinations of addition, subtraction, multiplication and division, and they will agree with his emphasis on the need for remedial tuition that involves more than superficial prescription of additional drill. Growth in arithmetical knowledge can certainly not be achieved unless a teacher succeeds in making children intelligent about the number system. To the emphasis this book puts upon the understanding of groups and their arrangements it is necessary, however, also to add a sensitiveness to pupils' difficulties and an appreciation of individual differences of which there is here little evidence. C. M. F.

Business Statistics. By GEORGE R. DAVIES and DALE YODER. Pp. viii, 548. 17s. 6d. 1937. (Chapman and Hall)

This book follows in part the traditional style of the American textbook of non-mathematical statistics, the emphasis of appeal being to the business statistician. Where it differs is in the addition of significance tests based on exact random sampling distributions, tests which are more reliable than the older ones based upon furnishing a probable error for each calculated constant. The authors are unwilling to let the older methods go altogether, for we get the usual lengthy lists of probable errors, but side by side with these, or sometimes at a later stage in the book, we get the more up-to-date tests. For example, in the case of the correlation coefficient, and later also for the multiple correlation coefficient and the correlation ratio, the usual test is given, and the authors then add that " the question of the reliability of the coefficient may also be approached by use of tabular or graphic summaries of significance. . . . These devices are derived by mathematical calculation and are based on what is known as a z-distribution ". In this connection, the F-table of Snedecor is reproduced in the appendix, and a number of charts enable the usual tests of significance to be carried out expeditiously.

The first part deals with collection and presentation of statistical data, averages and dispersion, index numbers, business trends and seasonal variations and cycles. For a textbook covering so many pages, the contents page is not altogether an adequate description of the subject matter, but an index at the end helps to identify the various branches of study. The reading is a little dull, except for those who revel in figures, but there is little of importance that does not find a place. The material is profusely illustrated by examples, and a collection of these at the ends of the chapters provides much illustrative material for the student reader. The second part covers simple correlation, multiple and partial correlation, curvilinear correlation, reliability and significance, which comes towards the end according to the old tradition that the methods involved are difficult, and should therefore be left until the student

* New York: Bureau of Publications, Teachers' College, Columbia University. 1935.

has mastered the straight-forward processes. Finally there is a chapter on the field of business statistics. An appendix contains many explanatory notes on calculation and other processes, as well as the tables and charts to which reference has already been made, and there is a bibliography. It ought to be added that there is a short treatment of the analysis of variance, which is a welcome innovation, helping to establish this branch of the subject as a permanent addition to the standard subject matter of such textbooks.

The book should fulfil well its function as a standard treatise on the methodology of the subject. J. W.

Mathematics for Electrical Students. 2nd Edition. Pp. vii, 225. 8s. 1938.

Mathematics for Shop and Drawing Students. 2nd Edition. Pp. vii, 225. 8s. 1938.

Technical Mathematics. 2nd Edition. *I.* Pp. vii, 246. 6s. *II.* Pp. ix, 277. 7s. 6d. *III.* Pp. vii, 149. 6s. 1938. By H. M. KEAL and C. J. LEONARD. (John Wiley, New York ; Chapman and Hall)

These books, first published in 1922, now appear in a second edition. That they have stood the test of experience would appear from the fact that the authors have not found it necessary to make any radical changes.

The three volumes of Technical Mathematics deal respectively with Algebra Geometry, and Trigonometry, though the separation is not complete as the volumes are intended to be taken consecutively ; thus, algebra is not neglected throughout the last two volumes, and some trigonometry is introduced in the first two. The ground covered is roughly : equations and formulae, directed numbers, factors, fractions, ratio, proportion and variation, powers and roots, quadratic and simultaneous equations, graphs ; properties of triangles, parallels, circles, similar figures, elementary properties of lines and planes, and the mensuration of solid figures ; logarithms ; solution of triangles, and some trigonometrical identities and equations ; the slide rule. The books intended for Electrical and for Shop and Drawing Students give shortened courses covering much the same ground (the geometry is condensed to a single chapter giving a résumé of elementary facts and theorems which the student is expected to know), but with special chapters of applications appropriate to each class of students.

The most useful feature of the books is the large number of practical examples. Also, in the case of exercises where the main purpose is, say, practice in manipulation, the field from which the formulae are drawn is indicated, so that the student can immediately appreciate the wide range of subjects in which the facility he is gaining can be employed. Apart from this, there is nothing particularly noteworthy about the method of presentation, and it would probably need to be considerably supplemented by the teacher ; for example, in algebra, rules are frequently formulated without further preliminaries on the ground of a single example. There are also some points (apart from the somewhat excessive price) which would make one chary of recommending them for general use in class. Thus, in checking equations the authors invariably adopt a method reprobated in § 4.2 of the M.A. Report on the Teaching of Algebra. The use of a dot as a multiplication sign in combination with a decimal point is confusing, especially as the positions adopted to distinguish them are the reverse of those used in this country ; and the use of two letters for a single variable (as hp. for horse power) seems an unnecessary complication which might easily lead to misunderstanding by weaker students in manipulating formulae containing them. However, teachers of elementary classes in technical colleges might find them useful as a source of examples. W. H.

An Introduction to Riemannian Geometry and the Tensor Calculus. By C. E. WEATHERBURN. Pp. xi, 191. 12s. 6d. 1938. (Cambridge)

This book is a short introduction to Riemannian Geometry. After an introductory chapter recalling certain results in algebra and analysis which are constantly used in the sequel, the author proceeds in three chapters to develop the elements of the tensor calculus, the operations of raising and lowering suffices by means of fundamental symmetric tensor of the second rank, and the theory of covariant differentiation. Applications follow to the theory of curves in a Riemannian space, to geodesics, to parallelism of vectors, to congruences of curves and to orthogonal ennuples. In the next chapter the curvature tensor is introduced and the rest of the book consists of applications to hypersurfaces and general subspaces of a Riemannian space ; here the work is simplified by the use of a generalised form of covariant derivative which is most appropriate in this connection, when functions arise which involve the coordinates both in the subspace and in the enveloping manifold. The book ends with an interesting historical review of the subject and an extensive bibliography.

The work is clearly and concisely written, and should prove a useful introduction to a very important branch of modern geometry. The printing is excellent throughout. J. A. T.

A Treatise on the Analytical Dynamics of Particles and Rigid Bodies with an Introduction to the Problem of Three Bodies. By E. T. WHITTAKER. Pp. xiv, 456. 25s. 1937. (Cambridge)

Since this edition only differs from the previous one by the correction of some errors and the insertion of references to work published since 1927, Prof. Whittaker obviously considers that his book is in its final form. This is not altogether surprising, because classical dynamics has practically reached the stage when no more soluble problems exist, and recent progress has been confined to rather formal developments concerning the ergodic theorem. The fact that the book has reached its fourth edition shows that it has become the standard work on the topics with which it deals. A. H. W.

Contributions to the Calculus of Variations, 1933-37. Pp. vii, 566. 13s. 6d. 1937. (University of Chicago Press : Cambridge University Press)

This is the third volume in a series consisting of Chicago Ph.D. theses. For the last thirty years or so there has been a group of mathematicians at Chicago who have specialised in the calculus of variations. Within wide limits they know all that is known about the subject and these theses are therefore very valuable both as original contributions and as expositions of the various topics discussed.

Broadly speaking there are two approaches to the calculus of variations. One is based on the Euler differential equations for the extremals. This leads to a theory of fields of extremals, transversality and conjugate or focal points, in which the guiding intuitions are geometrical. In fact, if one makes the ordinary assumptions as to regularity one has Finsler's generalisation of Riemannian geometry, in which the value of the fundamental integral is interpreted as the length of the arc of integration. There are various lines of further development. Among other things one may subject the variations to certain conditions over and above the usual boundary conditions. In isoperimetric problems, for example, a certain integral is to be minimised subject to the condition that another integral, taken along the same curve, shall have a fixed value. In the problem of Lagrange the curve of integration is to satisfy certain finite, or differential equations. One may also weaken the initial

assumptions so as to admit, for example, the possibility of extremals with corners, or one may proceed to " the large " as Marston Morse has done. The other approach is by means of the Hamilton-Jacobi partial differential equation of which the extremals are the characteristic curves. Of course, these two are closely related, the relation being analogous to that between the representation of a physical disturbance by rays and by a wave.

Of the thirteen theses in this book eight have to do with the isoperimetric type of problem, the problem of Lagrange and the somewhat analogous problem of Bolza. In six of these the method is based on the Euler equations and in the other two on the Hamilton-Jacobi equation. There is an article on fields of extremals for multiple integrals in which an interesting use is made of the Hamilton-Jacobi method ; one on unilateral variations, in which the curve of integration is required to lie on one side of a fixed curve ; one on Marston Morse's calculus of variations in the large, in which extremals extending beyond the first conjugate point are considered ; one on the minima of functionals and one on focal points considered in terms of Finsler's geometry.

J. H. C. W.

Introduction to Mathematics. By H. R. COOLEY, D. GANS, M. KLINE and H. E. WAHLERT. Pp. xviii, 634. 10s. 6d. 1937. (Harrap)

This book has the very admirable object of showing the student the true character of the subject, and its relations to the sciences, the arts, philosophy and knowledge in general. It is described as a book that can be used for a survey course for 1st year college students, who might be expected to cover about three-quarters of it in a class meeting three times a week for a year. The whole field of mathematics, ancient and modern, is drawn upon, and it is suggested that a choice of topics should be made. Not only is the book comprehensive, but each chapter contains one or more sets of exercises and ends with a list of books for reference and suggestions for further study. So it is very well adapted for use by students who have a suitable library at their disposal and the time and inclination to explore for themselves. In this sense it is an excellent introduction to the subject.

But it may well be thought that the authors have made a mistake in attempting to provide, at least to some extent, an introduction in the more literal sense of the word, that is, an introduction for beginners. Many pages are occupied in explaining work that must surely be familiar even to 1st year students. We are shown at length how to use logarithms, how to solve problems by means of equations, how to " factor " polynomials, how to solve quadratic equations. And if college students have to be taught the use of " standard form ", it is questionable whether the best method is by means of " Rule II ", which reads : " *To express in ordinary positional notation the number which in scientific notation is* $M.10^n$ *where* M *is between* 1 *and* 10, *and* n *is a positive integer, move the decimal point* n *places to the right in* M. *If the given number is* $M.10^{-n}$ *where* M *is between* 1 *and* 10, *and* $-n$ *is a negative integer, move the decimal point* n *places to the left in* M." The book would serve its purpose better if it were shorter, and it could have been conveniently shortened by supposing that the student had already reached what to English readers may be described as " School Certificate Standard ".

The historical order of development has been followed, but the history of the subject has not been given the prominence it deserves in a book of this type. On the other hand the book is particularly good in indicating a wide range of applications. It is, moreover, remarkably comprehensive, having chapters on relativity, non-euclidean geometry and infinite classes, besides sections on probability, statistics and groups, and discussions on axioms and the relation of mathematics to logic, philosophy and the arts.

English readers will, of course, compare it with *Mathematics for the Million* and with Whitehead's *Introduction*. It must be said at once that from the point of view of the million it is a good deal less lively and less attractive than Hogben's work, but it covers a wider field. It is really more like a very much expanded version of Whitehead's book, but of course without Whitehead. (The account of the extension of the idea of number is dull reading compared with Whitehead's, besides being less complete.) It is a book that may well be recommended to individual readers who wish to know more and do more than Whitehead has shown them, and it might also be useful for class work with Sixth Form boys for whom mathematics is a subsidiary subject. E. H. L.

Elementary Mathematical Analysis. By T. HERBERG. Pp. v, 120. 3s. 6d. 1938. (D. C. Heath, Boston; Harrap)

The author has succeeded, within the space of not more than a hundred pages, in collecting together a veritable galaxy of results which, one feels, constitutes a very representative survey of that part of the subject affectionately christened "elementary mathematics".

The subject matter with which he deals consists of the elements of coordinate geometry (including conic sections), the differential calculus, the theory of algebraic equations, complex numbers, and an introduction to the theory of probability by means of permutations and combinations. Each chapter contains a selection of worked examples and most of the fundamental results in that particular branch of the subject with which it is concerned. The proofs are for the most part informal and often consist merely of numerical verifications.

The book should prove very helpful to those Sixth Form students who, having matriculated, are desirous of pursuing a further course of study for the purpose of obtaining Higher School certificates. A series of test papers is included at the end but it is to be regretted that no answers have been supplied. Another lamentable omission is the absence of any reference to the much utilised resources of the integral calculus. J. H. P.

Computation and Trigonometry. By H. J. GAY. Pp. vii, 231, xvi, 143. 10s. 1938. (Macmillan)

This book contains a concise account of the elements of plane and spherical trigonometry, together with an outline of the essential methods of computation by means of logarithms and other mathematical tables, and their application, in particular, to trigonometrical problems.

The author has been wise, one feels, in including a very useful chapter dealing with the purely geometrical properties of lines and planes, a clear conception of which is indispensable to the student who is desirous of learning spherical trigonometry, and who finds himself confronted, for the first time, with the inevitable difficulties that accompany the task of visualising problems in three dimensions.

Many useful examples are worked out in the text, as illustrations; and each chapter is provided with a large number of exercises for the student.

As a fitting conclusion to the book there is an excellent collection of tables, which should prove more than adequate, even for the most laborious of calculations. J. H. P.

A School Solid Geometry. By C. O. TUCKEY. Pp. viii, 80. 2s. 3d. 1938. (Christophers)

In the recent *Report on the Teaching of Geometry in Schools* stress is laid on the value of solid geometry, not only for its own sake, but also as making it easier

for the pupil to grasp the fundamentals of plane geometry. If this view be correct it follows that no longer may solid geometry be regarded as a Sixth Form subject, to be taken after some years of plane geometry; the two must start together and go side by side through the school mathematical course. Mr. Tuckey was chairman of the Report Committee and in this little book he has devised a course on the lines indicated. The contents are, briefly:

Chapter I. Construction of cubes, cuboids, prisms, tetrahedra, with examples on their sections by planes.
(This is intended as the very beginning of the school course in geometry.)

Chapter II. Angles between lines and planes, found by actual drawing. Height and distance problems in three dimensions.

Chapter III. Congruence of triangles in space and of tetrahedra. Loci in space. Mid-point theorem.
(To be taken with congruent triangles and properties of the parallelogram.)

Chapter IV. Solid angles. Construction of regular polyhedra from their nets. Inequalities.
(This chapter is not meant to follow chronologically. The contents are to be dealt with separately and wherever they best fit in with the work of the particular class.)

Chapter V. Area and volume of solids. Plan and elevation. Sections.
(To be taken with area of triangle and parallelogram.)

Chapter VI. Sphere, cylinder and cone.

Chapter VII. Sections by parallel planes. Similarity.
(To go with similar triangles.)

Now most teachers will agree that simple solids are more interesting and natural to the ordinary boy than abstract plane figures, and that the construction of such solids offers an excellent mode of approach to the facts of plane geometry. The general scheme of the course, too, of building up a series of spatial relationships *pari passu* with the corresponding work on the flat, will meet with little dissent. How often have we groaned over the boy, even in the Mathematical Sixth, whose visual sense stops short at two dimensions! At the same time, a glance at the above table of contents will suggest the obvious query, whether a boy will be assisted, *in the early stages*, by having to grapple with two distinct sets of theoretical difficulties whilst still learning to manipulate the various geometrical instruments. On this point the present reviewer parts company with Mr. Tuckey. He would prefer the initial sequence to be:

Chapter I. From the teacher's freehand sketch of a box, rabbit-hutch, or other simple solid, the boy constructs a model. (This introduces the use of ruler, set-square, protractor, as well as the idea of plan and elevation.)
Regular polygons, inscribed in a circle or on a given base. Ruler and compass designs. Polygon patterns. Simple ruler and compass constructions, bisecting a given angle, line, etc. Drawing a perpendicular to a line. Heights and distances in one plane, the boy making his own inclinometer.

Chapter II. Construction of regular polyhedra from their nets. (Boys find this easy and exceedingly interesting. Incidentally it utilises the polygon constructions already learnt.)

Chapter III. Properties of parallels, suggested by figures occurring in I and II.

Chapter IV. Construction of tetrahedra, prisms, pyramids to introduce congruence of triangles.

Such topics as skew lines, non-planar heights and distances, solid angles, would be left until the donkey-work of plane geometry has been mastered, certainly until the deductive stage has got well under way.

However, one does not wish to dogmatize, and individual teachers will no doubt adapt the subject matter to suit their particular views. It is not quite clear, by the way, whether the book is intended for the teacher only; as it stands, it appears to be a little difficult for class use. For the teacher it provides a stimulating and suggestive guide, the exercises are interesting and cover a wide field, and one may go much farther than Mr. Tuckey's modest hope that the course will not be found a waste of time—on the contrary, the study in parallel of plane and solid geometry, on the main lines of the book, cannot fail to strengthen materially the student's grasp of both.

Two small points of detail call for attention: on p. 17, the word "centre", without qualification, is used in connection with equilateral triangles, and Ex. 4 on p. 57 needs straightening out. H. L.

A Course in General Mathematics. By H. T. DAVIS. Pp. xi, 316. 10s. 6d. 1938. (Principia Press, Indiana; Williams and Norgate)

The course of mathematics laid out in this book has been designed by the author for students preparing to become teachers of arithmetic. It is intended not only to instruct in numerical and algebraical processes, but also to form a background for the appreciation of arithmetic in its relation to other mathematical subjects. The author begins by tracing the growth of the number system, and in Chapter 1 deals with such things as the duodecimal system, the binary scale, casting out nines; on p. 11 we find such questions as: divide 56247 by TOT correct to two duodecimal places. We return in Chapter 2 to something more familiar in the form of fractions, root extraction and contracted methods. About seventy pages are then devoted to simple algebraic processes—as far as the binomial theorem and simple determinants. Chapter 9 is concerned with present values, annuities, bonds and perpetuities. Permutations, probability and the theory of equations, including Descartes' rule and Horner's method, complete the algebra. Chapters 12 and 13 cover the ground of simple trigonometry as far as half-angle formulae. No doubt the author has in mind something more than teachers of arithmetic in elementary schools, for in Chapter 14 he devotes about fifteen pages to functions of statistics and explains standard deviation, normal frequency curve, correlation coefficient, line of regression and time series. Teachers of the type of arithmetic for which the author caters will perhaps find this the most interesting chapter in the book; it seems to leave rather much to the reader, but no doubt this is intended to be amplified in class.

The author brings his book to a close with mathematical recreations and a few facts about famous mathematicians. Throughout the book, he intersperses historical comment, for, as he quotes Dr. George Sarton in the preface, "without history, scientific knowledge may become culturally dangerous; combined with history, tempered with reverence, it will nourish the highest culture." V. N.

Shorter Advanced Trigonometry. By C. V. DURELL and A. ROBSON. Pp. xi, 172. 5s. 1938. (Bell)

This book is an abbreviated form of *Advanced Trigonometry* by the same authors. It is intended to meet the requirements of the various Higher Certificate examinations and the authors are entitled to feel, as they do, that it will be sufficient for the needs of most pupils.

The book begins with the properties of triangles and quadrilaterals and such topics as the nine-point circle and polar circle of a triangle and circumcircle of a quadrilateral are considered. The general solution of equations, sub-multiple angles, projection and elimination follow, to complete Part I, which may be bought separately for 2s. 6d.

Part II, which may be had separately for 3s., deals with the expansion of the circular functions, hyperbolic functions, complex numbers, De Moivre's theorem, roots of equations, and the trigonometric functions of the complex variable. The proofs are sometimes condensed, but wherever this may seem drastic, the authors give references such as : consult E.T. (*Elementary Trigonometry*, Durell and Wright), A.A. (*Advanced Algebra*, Durell and Robson), etc., where no doubt the missing links will be found. There is a further reference table which enables the Key (price 15s.) intended for the *Advanced Trigonometry* to be used for the *Shorter Advanced Trigonometry*. This Key will be a convenience to harassed teachers who have not the time to wrestle with problems, as well as to non-specialist teachers who find themselves responsible for the advanced mathematics class. The price is very reasonable and the ability and sincerity of the authors is sufficient guarantee that it is well worth it. As we have said, Part II is largely concerned with the symbol i, but it leaves us with the feeling that the presentation is not so direct and incisive as we should like. At bottom, the symbol i is an algebraic symbol with the property $i^2 = -1$, or an algebraic operator with the property $i^2 = -$; like Maclaurin's expansion or the index law, it is a channel down which all sorts of mathematics either flow naturally, or can be made to flow. It is in the realisation of the above brutally terse definition that we ask for gentle guidance. The authors lean heavily on that overworked word " corresponds ". Correspondence is synonymous with relationship, dependence—the technical name for which is functionality in its widest sense. When they say " the number $[a, b]$ corresponds to the point P, whose coordinates are (a, b) " they are simply saying that $P = f[a, b]$ and $P = F(a, b)$; we are not told anything about the nature of f and F, whether they are identical, and if not, what is the precise difference. This is the sort of thing teachers want to know. Later on, the need of some further correspondence is felt, for the section on addition begins by saying that P and Q are points corresponding to $[a, b]$ and $[c, d]$ and R corresponds to $[a+c, b+d]$ but ends by saying that the displacement corresponding to the sum of two complex numbers is the vector sum of the displacements corresponding to the two numbers. V. N.

Solid Mensuration. By W. F. KERN and J. R. BLAND. Second edition. Pp. vii, 172. 10s. 1938. (John Wiley and Sons, New York ; Chapman and Hall)

This second edition is more than double the size of the first edition, and is in many ways much better. The first edition, which was reviewed in 1935 in the February issue of the *Gazette*, covered the mensuration of the prism, pyramid, cylinder, cone and sphere to about School Certificate standard. The theorems of Pappus, and some theorems on the straight line and plane have been now added, but without proofs. This omission is intentional as the book is meant to be used as a supplement to an organised course in solid geometry ; students would never understand the definition of polyhedral angle which is given unless the concept had already been formed. The authors have altered their original book to the extent that the second edition is substantially a new book. They lay stress on the use of examples with a practical bias as a means of developing space intuition, and teachers of mensuration in technical schools should find in this book enough examples of this type to meet their needs. V. N.

Funktionentafeln mit Formeln und Kurven. (Tables of functions with formulae and curves.) By E. JAHNKE and F. EMDE. 3rd edition. Pp. xii, 305. RM. 11.25. 1938. (Teubner)

"The indispensable aid of the mathematician, physicist and engineer." The publisher's phrase is one which a reviewer can quote without fear, and he can add that the teacher who has occasion to mention or use some of the more advanced functions of mathematics with technical classes will find this volume of considerable value in providing numerical and graphical material. The excellent "relief" diagrams are also most useful in helping to make these functions concrete realities instead of mathematical abstractions; a relief diagram here means a picture of a surface whose ordinate at the point z in the Argand plane is $|f(z)|$.

The text is in German and English, as in the second edition. The volume contains about 300 pages as against the 330 of the second edition, but room has been found for additions by the omission of the first 75 pages of the second edition which dealt with the elementary functions. This part is to be issued shortly as a separate volume. The main sections deal with the logarithm-integral, the Gamma function, the error integral, Theta functions, elliptic functions and integrals, Legendre functions, Bessel functions, the Riemann Zeta-function, and additions on the confluent hypergeometric functions and Mathieu functions. The new edition also corrects errors and extends some of the original sections.

Apart from the somewhat unattractive fount of heavy, equal-height figures used in the tables, the book is admirably produced. Excellent indexing makes reference easy, and the diagrams deserve very high praise. T. A. A. B.

Necéssaire Mathématique. By M. CURIE and M. PROST. Pp. 116. 20 fr. 1937. Actualités scientifiques et industrielles, 502. (Hermann, Paris)

This little book was written primarily to satisfy the mathematical needs of students wishing to qualify for the Certificat d'Etudes Physiques, Chimiques et Biologiques, or for the Certificat d'Etudes Supérieures Physiques, Chimiques et Naturelles. The title is a novel one, and the authors have striven not to exceed the limitations which it imposes on them. The result of this is a most refreshing brevity. As is stated in the preface, appeal is made to intuition rather than to a more rigorous form of reasoning, which makes the book easy to read.

The subject matter covered includes algebra and trigonometry. A chapter on graphical representation of functions leads up to the chapter on differential calculus, which concludes with a brief reference to Maclaurin's theorem. The logarithmic and exponential functions are treated fairly fully in the next chapter. The chapter on integral calculus is probably the best in the book, and linear differential equations of the first and second orders receive adequate treatment. A novel feature is a short appendix on probability.

Considerable use is made of differentials, which would have been a praiseworthy feature had these been adequately defined. It is phrases like "On appelle infiniment petit un nombre plus petit que le plus petit nombre fini qu'on puisse imaginer" that do much to create in the mind of the layman the impression that the calculus is a form of mysterious jugglery. It is surely not difficult to find a logically sounder definition suitable even for a book of this type.

The illustrative examples, which are numerous, are usually practical in character, and have been well chosen. The text is also illustrated liberally by means of well drawn diagrams. There are however one or two misprints which might perplex the unsuspecting reader.

Unfortunately the type of student in this country to whom this book might have been invaluable will probably be deterred by the French language. The book should however serve as a useful guide to anyone who is engaged in teaching this type of mathematics. B. M. B.

La durée extrême de la vie humaine. By E. J. GUMBEL. Pp. 65. 18 fr. 1937. Actualités scientifiques et industrielles, 520; statistique mathématique, II. (Hermann, Paris)

During recent years there has been an increasing interest about the duration of human life and the possibility of a continuation of the improvement that has taken place owing, it is assumed, to an increase in our sanitary and medical knowledge. Many formulae have been constructed to express the life curve, the best known and most used being those by Lexis and what is known as the Gompertz-Makeham curve, which are the formulae chiefly used by M. Gumbel in estimating the final duration of human life. According to these formulae, the age of Methuselah (969 years), although extremely improbable, would not be impossible.

M. Gumbel has also made estimates from data given by some American tables, giving records for Blacks and Whites, native and foreign, and also from Swedish and Swiss statistics. For Sweden he gives as the final age for the year 1932 as 105·79, and estimates the final age for the year 2036 as 106·49. Amongst American records we find the oldest male was a negro, 135; the oldest white, 119. The chief difficulty in this matter is the absolute unreliability of the records. We have in this country rare cases, for we find according to documents printed in the Royal Society *Transactions* the case of Thomas Parr, who is said to have lived 152 years 9 months, and Henry Jenkins, 169 years. It is difficult to see how reliable records could be obtained for any period over 100 years, and all such records must be suspect. The paper by M. Gumbel is well worth the attention of all those interested in statistical questions. W. S.

1210. The strategy of revolution is much more complex and more dynamic than the strategy of military or naval war. Even in the extremely simplified presentation given above, it should be clear that the strategy of revolution compared to the strategy of ordinary warfare is as mathematics compared to arithmetic.—R. Page Arnot, *A Short History of the Russian Revolution*, I. [Per Mr. G. L. Beach.]

1211. Capt. G. E. T. Eyston's 6,000 H.P. monster car, Thunderbolt, is home from the Bonneville Salt Flats, Utah, where it made the land speed record of 312·20 M.P.H.

The horse-power produced by Thunderbolt in one minute would drive an ordinary 10 H.P. saloon car from London to Manchester—200 miles—at 30 M.P.H.—*Daily Herald*, 29th December, 1937. [Per Mr. C. W. Bartram.]

1212. Peignot discusses, by the way, with his usual scientific precision, as a department in Bibliography, "Titres de livres qui ont introduit en erreur des Bibliothécaires et des Libraires peu instruits". After mentioning a treatise, *De Misses Dominicus*, which was not a religious book, . . . he comes nearer our own door in telling how "un ignorant avait placé le *Traité des Fluxions de Maclaurin* avec les livres de pathologie, prenant pour une maladie les fluxions mathématiques".—J. Hill Burton, *The Book Hunter*, p. 126. [Per Mr. J. B. Bretherton.]

MATHEMATICAL FILMS.

THE following reviews of Mathematical Films are reproduced from the *Monthly Film Bulletin* of the British Film Institute by permission of the Institute.

EQUATION $\ddot{X}+X=A \sin Nt$, THE (Great Britain, 1937)

35 mm. Silent (flam.). 16 mm. Silent (non-flam.). 750 ft. 12 mins. 1 reel.

Production : Notation by Robert Fairthorne ; animation by B. G. D. Salt.

Description : Cine-diagrams.

Purpose : Mathematics teaching.

Teaching Notes : Available.

Distributors : B. G. D. Salt, 5 Carlingford Road, London, N.W. 3 (35 mm. and 16 mm.). National Film Library, British Film Institute, 4 Great Russell Street, London, W.C. 1 (16 mm.).

Conditions of Supply : B. G. D. Salt : 35 mm. and 16 mm. loaned on hire at 2s. for three days. National Film Library : hiring terms on application.

Contents : This film is an elaboration of the method of representation used in the earlier film $\ddot{X}+X=0$. The lines representing the acceleration and displacement now bear against the opposite arms of a balance. Instead of being fixed (which would correspond to the case of $\ddot{X}+X=0$) the pivot is given a harmonic displacement by means of an eccentric, so that the behaviour of the system is governed by the relation $\ddot{X}+X=A \sin (Nt)$. The system is shown in motion for several cycles, care being taken to show the effect of the external force being " in " or " out " of step with the natural motion of the system. The moving diagram also shows that the motion is a superimposition of two S.H.M.'s. By further elaboration of the diagram it is possible to show variations in the energy according to the variations of the constant of the equation.

Appraisal : Even more clearly than in the earlier film is revealed here the difficulties inherent in any attempt to transform really complicated mathematical laws into visual terms. The difficulties of understanding the nature of the mechanism in this case are of the same order as those involved in understanding the mathematical theory itself. For students whose grasp of formal relations is defective this kind of translation into a mechanical model is no doubt of use, and for all students the attempt to follow such a mechanism is undoubtedly instructive. This is essentially a film which needs to be shown several times, preferably with breaks, before its principles can be understood. Used judiciously, with ample commentary from a teacher who himself understands it, it might prove a valuable basis for the discussion of general features of the differential equations involved.

Suitability : Advanced students of pure or applied mathematics in first or subsequent years of the university course. Its value in schools would be restricted to the occasional case of the unusually gifted mathematician.

MOUVEMENTS VIBRATOIRES (France, 1933)

35 mm. Silent (flam.). 16 mm. Silent (non-flam.). 1000 ft. 11 mins. 1 reel.

Production : Atlantic Films.

Description : Cine-diagrams, with French sub-titles.

Purpose : Teaching.

Teaching Notes : Not available.

Distributors : Film Centre Ltd., 34 Soho Square, London, W. 1.

Conditions of Supply : Loaned on hire : 35 mm. version £1 ; 16 mm. version 7s. 6d.

Contents : An analysis of the formation of a sine curve by a pendulum. The effect of combining the movements of two pendulums producing sine curves of differing frequency. Analysis of a vibrating cord.

Appraisal : This is a delightful example of the help the film can give to mathematical instruction. The relation between simple harmonic motion, uniform motion in a circle and the sine curve are shown by means of moving diagrams which not only demonstrate the mathematical points but provide considerable aesthetic pleasure. This is one of many cases where perception of the motion of curves produces conviction in a way which no static blackboard diagram can achieve. The superimposition of a travelling sine curve to produce stationary waves shows these qualities of lucidity and beauty to an exceptional degree. The mathematical argument is strengthened, in a most welcome way, by shots of ingenious physical apparatus. Sceptics as to the value of the film in mathematical teaching might well be invited to see this film ; they could hardly fail to be both impressed and charmed.

Suitability : For sixth form pupils, who should incidentally have no difficulty in understanding the French captions.

FORCE OF GRAVITY (France, 1935)

16 mm. Silent (non-flam.). 368 ft. 15 mins. 1 reel.

Production : Benoit-Levy, Paris.

Description : Direct photography, slow-motion photography, cine-diagrams and captions.

Purpose : Teaching.

Teaching Notes : Not available.

Distributors : Educational and General Services Ltd., 37 Golden Square, London, W. 1.

Distributors' Cat. Ref. : J. 113.

Conditions of Supply : Loaned on hire at 2s. 6d. per day and 1s. for each additional day. Sale price on application.

Contents : A comparison is made between the rates of falling bodies of different masses through air and in a vacuum. A diagrammatic explanation is given of the apparatus used for accurate measurement of the acceleration of a body falling freely under gravity, and three sets of measurements are taken to enable the student to calculate and compare the various values of s/d^2.

Appraisal : This is a very useful film. The experiment described is one which could not easily be performed in schools. The arrangement of the apparatus (an electrical version of Attwood's machine) is on the whole very clearly demonstrated, admirable use being made of various devices for attracting attention to the significant points in the complicated diagram. The film might well have allotted more time to the explanation of the apparatus and a little less to the calculation of simple results. The three numerical values are insufficient for any general conclusions as to the gravitational law. If these defects were remedied the film would be excellent of its kind.

Suitability : For the elementary teaching of mechanics (Ages 12-14).

1213. Thadeus Kosciusko was born in 1755, of a poor but noble family, and received the first elements of his education in the corps of cadets at Warsaw. There he was early distinguished by his diligence, ability, and progress in mathematical science, insomuch that he was selected as one of the four students chosen at that institution to travel at the expense of the state.—Alison, *History of Europe*, III, p. 189, footnote.

1214. Higher mathematics and physics have reached a stage of such intellectual differentiation that soon it will become impossible to refer to them as to rational sciences, and they might find themselves one day in the unusual company of such a scientific bastard as occultism.—Rom Landau, *God is My Adventure*, p. 8. [Per Mr. A. F. Mackenzie.]

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